A Novel Application of the Signless Laplacian Matrix to Identify Bipartite Subgraphs

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CERTIFICATE

This is to certify that **Debdas Paul** (Registration No. 108427 of 2009-2010, Class Roll No. of 000910502028), student of Computer Science and Engineering, Jadavpur University, has done a thesis under my supervision titled ***A Novel Application of Signless Laplacian Matrix to Identify Bipartite Subgraphs***. The thesis is approved for submission towards partial fulfilment of the requirements for the degree of **Master of Computer Science and Engineering (MCSE)** under Jadavpur University for the session 2010-2011.

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To... Debaprasad Mukherjee... who transforms my ideas.

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Abstract

This work is an exploration of a practical application of the newly born Spectral Graph Theory based on the Signless Laplacian Matrix (also called the Q-theory) by establishing some new theorems regarding bipartite graphs. Bipartite graphs are well known for their importance in modelling real world networks starting from protein-protein interaction network to the world-wide web. For example, bipartite components of a protein interaction networks comprise those node sets which are involved in interaction according to the *Lock and Key model* based on complementary binding domain.

For a connected simple graph G, the smallest eigenvalue of its signless Laplacian spectrum indicates whether or not G is bipartite, while the largest eigenvalue of its normalised Laplacian spectrum indicates the same also. In this thesis, some tight inequalities relating the smallest signless Laplacian eigenvalue to the largest normalised Laplacian eigenvalue are derived. It is also investigated how vectors yielding small values of the Rayleigh quotient for the signless Laplacian matrix can in general be used to identify certain subgraphs. Moreover a sufficient condition under which these subgraphs are in general guaranteed to be bipartite is also provided. As these subgraphs are expected to be "weakly connected" to the rest of the graph, sufficient conditions for that are also provided. All these relations derived under this work are verified by applying them on some graphs with degree sequences approximately following a power law degree distribution with exponent 2.1 thus forming a scale-free network such as a protein-protein interaction network and this method results into exploration of more number of bipartite subgraphs compared to those obtained through adjacency matrix eigenvector method. Hence it provides better results.

Keywords: *Bipartite subgraph, Signless Laplacian matrix, Normalized Laplacian matrix.*

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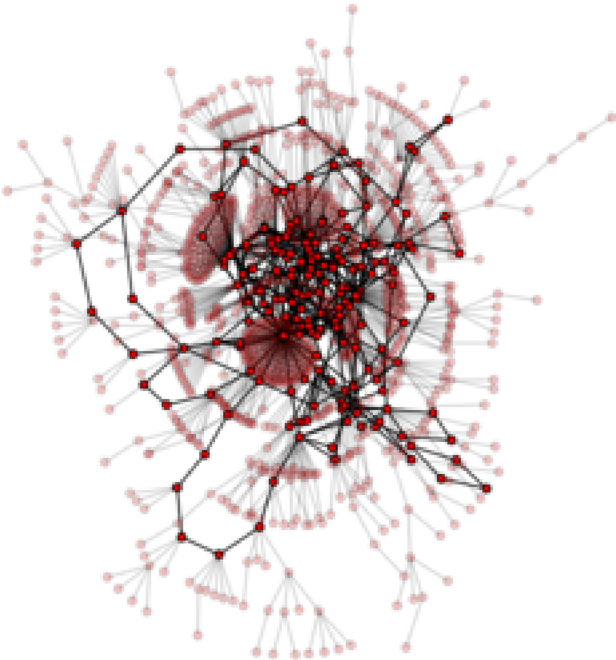

Chapter 1

Introduction

1.1 Spectral Graph Theory- an overview

Based on linear algebra and well-developed theory of matrices, **Spectral Graph theory** is concerned with the relationship between the algebraic properties of the spectra of certain matrices associated with a graph and topological properties of that graph. Although based on theory of matrices, this theory has its own reasoning and characteristic features for which it can't be reduced to the theory of matrices. In the next few subsections, we will be familiar with this fact.

1.1.1 The M-Theory

[1] As in spectral graph theory, we study a graph by means of the eigenvalues of the some graph matrix M, so it is also called *M - Theory* (Cvetkovic & Simic (2009)). Depending upon M-the theory changes. So from that point of view the spectral theory of graph is not unique. Three major matrices are: A- the *Adjacency matrix*, L- the *Laplacian matrix*, and Q- the *Signless Laplacian matrix*. Based on these three matrices three different spectral theory has been developed which are called *the A-theory, the L-theory and the Q-theory* respectively. The aim of spectral theory of graph is to study these theories and their interactions.

[1] This M-theory should not be confused with that in theoretical physics which is an extension of string theory with 11 dimension

[1]Let Φ be a graph without multiple edges. The adjacency matrix of Φ is a 0-1 matrix A of vertex set VΦ of Φ, where $A_{xy} = 1$ when there is an edge from x to y in Φ and $A_{xy} = 0$ otherwise. For multi graphs (possibly with loops) in which case A_{xy} equals the number of edges from x to y. Throughout the thesis , we will concentrate only on undirected graphs.
Let Φ be an undirected graph without loops. The (vertex-edge) *incidence matrix* of Φ is the 0-1 matrix M, with rows indexed by the vertices and columns indexed by the edges, where $M_{xe} = 1$ when vertex x is an endpoint of edge e.
Let Φ be a directed graph without loops. The directed incidence matrix of Φ is the 0-1 matrix N, with rows indexed by the vertices and columns by the edges, where $N_{xe} = -1, 1, 0$ when x is the head of e, the tail of e, or not on e,respectively.
Let Φ be an undirected graph without loops. The *Laplace matrix* of Φ is the matrix L indexed by the vertex set of Φ, with zero row sums, where $L_{xy} = A_{xy}$ for $x \neq y$. If D is the diagonal matrix, indexed by the vertex set of Φ such that D_{xx} is the degree (valency) of x, then $L = D - A$. The matrix $Q = D + A$ is called the *Signless Laplace* matrix of Φ (we will discuss the spectral theory based on the signless Laplacian matrix in a short while). An important property of the *Laplace* matrix L and the *Signless Laplace* matrix Q is that they are positive semi-definite. Indeed, one has $Q = MM^T$ and $L = NN^T$ if M is the incidence matrix of Φ and N the directed incidence matrix of the directed graph obtained by orienting the edges of Φ in an arbitrary way. It follows that for any vector u one has $u^T Lu{=}\Sigma_{xy} \, (\, u_x - u_y \,) \, ^2$ and $u^T Qu{=}\Sigma_{xy} \, (\, u_x + u_y \,) \, ^2$, where the sum is over the edges of Φ.

1.1.2 The Spectrum of a Graph

The (ordinary) spectrum of a finite graph can be defined as the set of eigenvalues of A together with their multiplicities. The Laplace spectrum of a finite undirected graph without loops is the spectrum of the Laplace matrix L. The rows and columns of a matrix of order n are numbered from 1 to n, while A is indexed

[1]These informations and the informations just in the next subsection are mostly taken from (Brouwer & Haemers) with slight modifications in few places. One can consult any other references as these are very common.

by the vertices of , so that writing down A requires one to assign some numbering to the vertices. However, the spectrum of the matrix obtained does not depend on the numbering chosen. It is the spectrum of the linear transformation A on the vector space K^X of maps from X into K, where X is the vertex set, and K is some field such as $\mathbb{R}$ or $\mathbb{C}$. The characteristic polynomial of is that of A, that is, the polynomial $_{\kappa A}$ defined by $_{\kappa A}\ (\theta) = det\ (\ \theta\mathrm{I} - A\ )$.

Example Let be the path P_3 with three vertices and two edges. Assigning some arbitrary order to the three vertices of , we find that the adjacency matrix A becomes one of

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} or \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} or \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The characteristics polynomial is $_{pA}\ (\theta) = \theta^3 - 2\theta$. The spectrum is $\sqrt{2}$, 0 ,$-\sqrt{2}$. The eigenvectors are :

$$\begin{bmatrix} \sqrt{2} & 2 & -\sqrt{2} \end{bmatrix}^T and \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T and \begin{bmatrix} \sqrt{2} & -2 & \sqrt{2} \end{bmatrix}^T.$$

Each entries in a eigenvector corresponding to a vertex in the graph. For example in the first eigenvector the similar entries correspond to two end vertices and rest one correspond to the middle vertex,in the second eigenvector 1 and -1 correspond to two end vertices and 0 corresponds to the middle vertex. Now mathematically ,let for an eigenvector u, we write u_x as a label at the vertex x; we have $Au = \theta u$ if and only if $\Sigma_{y \leftarrow x}\ u_y = \theta u_x$ for all x.) The Laplace matrix L is one of

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} or \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} or \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Eigenvalues are 0 , 1 and 3. The Laplace eigenvectors are

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T and \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T and \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^T.$$

We have $Lu = \theta u$ if and only if $\Sigma_{y \sim x}\ u_y = (d_x\text{-}\theta)$ for all x, where d_x is the degree of vertex x.)

Example Let be a directed triangle with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The A has characteristics polynomial $\kappa_A(\theta) = \theta^3$ - 1 and spectrum 1, ω, ω^2, where ω is a primitive cube root of unity.

Example Let be the directed graph with two vertices and a single directed 0 1 edge. Then A=$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $\kappa_A(\theta) = \theta^2$. So A has eigenvalue with geometric multiplicity (i.e., the dimension of the corresponding eigenspace) equal to 1 and algebraic multiplicity (i.e., its multiplicity as root of polynomial κ_A) equal to 2.

1.2 Q- theory

[1] The Q-theory or the spectral graph theory based on the signless Laplacian matrix was first developed by Prof. Dragos Cvetkovic of faculty of Electrical Engineering, University of Belgrade, Serbia in the year 2005. This theory has been developed due to having less *spectral uncertainty* (see A.2) while studying graphs compare to two other theories i.e, *A -theory - spectral Graph theory based on the Adjacency matrix*, and *L-theory- spectral graph theory based on Laplacian matrix.* From the year 2005 onwards numerous developments on this theory with several new conjectures have been made. We will focus especially on the theorems for bipartite graphs based on signless Laplacian matrix as it is the main theme of the this work. But before that let's go through some preliminaries.

According to a comprehensive research report by Prof. Cvetkovic (Cvetkovic (2010)) , apart from the issue of spectral uncertainty , there are two other reasons for which Prof. Cvetkovic found possibilities to formulate this theory - An existing connection between Q-eigenvalues and spectra of line graphs implying that the existing developed theory of graphs with least eigenvalues - 2 (Cvetkovic

[1]This Q comes from the word Quasi -laplacian matrix which is the alternative name of the signless laplacian matrix

(2005)),(Cvetkovic *et al.* (2004)) and a pragmatic reason - it was a very nascent field that time.

1.2.1 Some important properties of Q-spectra with respect to bipartite graphs

Consider a graph $G(n, m)$[1] and R be its *vertex-edge incidence matrix* . Then the following relations can be made:

$$RR^T = D + A, R^T R = A(L(G)) + 2I, \tag{1.1}$$

where A(L(G)) is the adjacency matrix of L(G), the line graph (see Appdx.A) of G. Since the non -zero entries in RR$^T and R^T R$ are the same, we immediately get that

$$P_{L(G)}(x) = (x + 2)^{m-n} Q_G(x + 2). \tag{1.2}$$

These two equations are very common and can be found to any book or paper (Cvetkovic (2010)) related to this. From (1.1), it is evident that the signless Laplacian is a *positive semi-definite* matrix i.e., all its eigenvalues are non-negative. Let the Q-eigenvalues q$_1 \geq q_2 \geq .... \geq q_n$. The largest eigenvalue q_1 is called Q-index of G.

When applying the Perron-Frobenius theory of non-negative matrices (see A.1) to the signless Laplacian Q, the results obtained are same as that of the adjacency matrix. In a connected graph the largest eigenvalue is simple with a positive eigenvector. For a connected graph $q_1(G) > q_1^{'}(S(G))$, where $S(G)$ is a subgraph of G.

Now the smallest eigenvalue of the signless Laplacian has a special importance with respect to bipartite graphs. Here we will go through some propositions and

[1] G has no isolated vertices

theorems regarding the same. These prorpostions are very common. These can be found in (Cvetkovic (2010)).

Proposition 1.2.1. *The least eigenvalue of the signless Laplacian of a connected graph is equal to* 0 *if and only if the graph is bipartite. In this case* 0 *is a simple eigenvalue.*

Proof. Let $x^T = (x_1, x_2, ..., x_n)$. For a non-zero vector x we have $Qx = 0$ if and only if $R^T x = 0$. The later holds if and only if $x_i = x_j$ for every edge, i.e. if and only if G is bipartite. Since the graph is connected, x is determined up to a scalar multiple by the value of its coordinate corresponding to any fixed vertex i. $\square$

Remark 1.2.1. *According to Theorem 2.2.4 in (Cvetkovic et al. (2004)), the multiplicity of the eigenvalue* -2 *in* $L(G)$ *is equal to* $m - n + 1$ *if* G *is bipartite ,and equal to* $m - n$ *if* G *is not bipartite. This together with formula (1.2) yields the assertion of the proposition.*

Corollary 1.2.1. *In an graph the multiplicity of the eigenvalue 0 of the signless Laplacian is equal to the number of bipartite components.*

Proposition 1.2.2. *The eigenspace of the* Q*-eigenvalue* 0 *of a graph* G *determines sets of vertices and the bipartitions in bipartite components of* G*.*

Proof. Let $x^T = (x_1, x_2, ..., x_n)$.For a non-zero vector x we have $Qx = 0$ if and only if $x_i = x_j$ for every edge. If the graph is connected (and then necessarily bipartite) , x is determined up to a scalar multiple by the value of its coordinate corresponding to any fixed vertex i. If G is disconnected, at least one component is bipartite. If a vertex i belongs to a non-bipartite component, then $x_i = 0$. Using Corollary 1.2.1 we determine the number of bipartite components as the multiplicity of eigenvalue 0. For each bipartite component we have an eigenvector with non-zero coordinates exactly for vertices in this component. Now, vertex sets of bipartite components are determined by non-zero coordinates in vectors of a suitably chosen orthogonal basis of the eigenspace of 0. The sign of these coordinates determines colour classes within bipartite components. $\square$

The least eigenvalue of Q-spectra also helpful in determining the measure of non-bipartiteness of a graph(Desai & Rao (1994)). In particular for a connected graph,

$$\frac{\psi^2}{4d_{max}} \leq q_n \leq 4\psi,$$

Where d_{max} is the maximal vertex degree and ψ is the measure of non-bipartiteness which is minimum over all non-empty proper subset S of $V(G)$ of the quotient

$$\frac{|cut(S)| + e_{min}(S)}{|S|}.$$

where the numerator is the minimum number of edges whose removal from edge set $E(G)$ disconnects S from $V - S$ and results in a bipartite subgraph induced by S (Desai & Rao (1994)).

Remark 1.2.2. *Q-polynomial is only help full in giving information about bipartiteness of a graph when we know that the graph is connected and on the other hand it is also interesting that Q-polynomial does not tell us about connectedness of graph (Cvetkovic (2010)). As we know that when a graph is bipartite the Q-polynomial and the L-polynomial are same and as L polynomial is the indication of connectedness of a graph then, for Q -polynomial, if we know one of the two conditions i.e., connectedness and bipartiteness, we can easily reconstruct one from another. Next we are going to see an important proposition regarding the above.*

Proposition 1.2.3. *The Q-polynomial of a graph is equal to the characteristics polynomial of the Laplacian if and only if the graph is bipartite.*

Proof. Suppose that the graph G is bipartite, with parts U and V. Consider the determinant defining $Q_G(x)$. Multiply by 1 all rows corresponding to vertices in U and then do the same with the corresponding columns. The transformed determinant now defines the characteristic polynomial of the Laplacian of G. The multiplicity of the eigenvalue 0 in the Laplacian spectrum is equal to the number of components, while for the signless Laplacian, the multiplicity of 0 is equal to the number of bipartite components. Therefore in non-bipartite graphs the two polynomials cannot coincide. $\square$

. As we can not establish whether a graph is bipartite or not from its Q-polynomial, we do not know whether $Q_G(\lambda)$ really equals $L_G(\lambda)$. Therefore along with the connectedness information one should mention the number of connected components to overcome the limitation in use of the above proposition.

1.3 Objective & overview of the work

Let G be a graph with adjacency matrix A, and denote the diagonal matrix of vertex degrees for G by D. The matrix $Q = D + A$ is known as the *Signless Laplacian* matrix for G, and has been the subject of a flurry of recent papers. The surveys (Cvetkovic & Simic (2009)), (Cvetkovic & Simic (2010a)), (Cvetkovic & Simic (2010b)) give an overview of the research on the *Signless Laplacian* matrix from the perspective of spectral Graph theory. In particular, it is known that the signless Laplacian matrix Q for a graph G is positive semi-definite, and that the multiplicity of 0 as an eigenvalue of Q coincides with the number of connected components of G that are bipartite (Cvetkovic (2005)); thus, for a connected graph G, the smallest eigenvalue of Q is positive if and only if G is not bipartite. In a similar vein, if G is a Graph with no isolated vertices, the matrix $\mathcal{L}$ $= I - D^{-\frac{1}{2}}AD^{\frac{1}{2}}$ is known as the *Normalised Laplacian* matrix for G (here, A and D are as above). The spectral properties of the normalised Laplacian matrix are also well-studied, and (Chung (1997)) gives an extensive discussion of how the spectral properties of $\mathcal{L}$ reflect the structure of G. In particular, it is known that the eigenvalues of the *normalised Laplacian* matrix fall in the interval $[0, 2]$, and that the multiplicity of 2 as an eigenvalue of Q coincides with the number of connected components of G that are bipartite. Thus, as above, for a connected graph G, the largest eigenvalue of $\mathcal{L}$ is less than 2 if and only if G is not bipartite. Suppose now that we have a Graph G with *Signless Laplacian* matrix Q and *Normalised Laplacian* matrix $\mathcal{L}$. In view of the observations above, it might be interpreted that the smallest eigenvalue of Q, say , as a measure of how bipartite our graph G is; alternatively, it is also interpreted the largest eigenvalue of L, say as a measure of how bipartite our Graph G is. (It should be noted in passing that both interpretations are philosophically related to the work of Fielder (Fiedler (1973)), who proposed that the second smallest eigenvalue of the Laplacian matrix $D - A$ be used as a measure of the connectivity of G.) Since both and provide notions of how bipartite G is, it is natural to investigate the relationship between these two quantities. We do so in section 2.1, providing upper and lower bounds on in terms of ; both bounds are tight, and we give examples of infinite families of non-bipartite, non-regular graphs for which the upper and lower bounds are attained, respectively.

The discussion above gives rise to the following scenario. Suppose that our connected graph G is not bipartite, but its smallest *Signless Laplacian* eigenvalue is close to zero. One might then have the intuition that G contains a bipartite sub graph that is not very well connected with the rest of the graph. How might one can identify such a sub graph? In section 2.2 , a condition (based on a *Rayleigh quotients*(see A.4)) that is sufficient to identify a bipartite sub graph H in section 2.2 is provided. it is also given a condition at the same under which the number of vertices of the sub graph H having at least one neighbour in $G \backslash H$ can be bounded. Both results serve to reinforce the intuition noted above.

As a part of interest in identifying bipartite sub graphs that are only weakly connected to the rest of the graph stems from the study of complex networks such as protein-protein interaction networks, the world wide web, and certain social networks (see (Newman (2003)) for a survey of work on this topic). Within such networks, bipartite structures are important in verifying some useful properties (Guillaume & Latapy (2004)), (Thomas *et al.* (2003)) . For example, in the case of protein-protein interaction networks, these bipartite sub graphs represent biologically relevant interaction motifs (Morrison *et al.* (2006)). In chapter 3, we use eigenvectors of the *signless Laplacian* matrix to generate certain *Rayleigh quotients* that enable us to identify bipartite sub graphs using the results of section 2.2. That approach is applied to randomly generated graphs that approximate scale-free networks (for detailed computation results see Table C.2 and Table C.4) , and to a particular protein-protein interaction network (see Table C.4 and Table C.5).

Chapter 2

Bipartite Subgraphs and the Signless Laplacian Matrix

[1]

2.1 Extreme eigenvalues for the signless and normalised Laplacian matrices

For a connected graph G, both the smallest signless Laplacian eigenvalue and the largest normalised Laplacian eigenvalue for G serve as indicators as to whether or not G is bipartite. Since both of these eigenvalues identify a common feature of G (i.e., bipartiteness, or the lack thereof) it is natural to seek a quantifiable relationship between these two eigenvalues. The following result does precisely that.

Theorem 2.1.1. *Let G be a connected graph on n vertices with signless Laplacian matrix Q and normalised Laplacian $\mathcal{L}$. Let the smallest eigenvalue of Q be μ and the largest eigenvalue of $\mathcal{L}$ be λ. Denote the maximum and minimum degrees of G by Δ and δ respectively. Then*

[1] This chapter is based upon the work done in collaboration with Prof. Steve Kirkland of National University of Ireland, Maynooth, Ireland, Europe. This work has been presented in The workshop on *Linear Algebraic Techniques in Combinatorics/Graph Theory*, Jan 30-Feb 04, 2011, at the Banff International Research Station, Alberta, Canada.

$$2 - \frac{\mu}{\delta} \leq \lambda \leq 2 - \frac{\mu}{\Delta}. \tag{2.1}$$

Proof. Let A denote the adjacency matrix of G, and D denote the diagonal matrix of vertex degrees , so that $Q = D + A$ and $\mathcal{L} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$.

We consider the left hand inequality in (2.1). Let x be an eigenvector of Q corresponding to μ. Without loss of generality, we write x as $\left[\begin{array}{c} x_1 \\ \hline x_2 \end{array}\right]$, where both x_1 and x_2 are nonnegative vectors. Partition D and A conformally with x as $\left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right]$ *and* $\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$. Since $x^T Q x = \mu x^T x$, we find that

$$x_1^T D_1 x_1 + x_1^T A_{11} x_1 - 2x_1^T A_{12} x_2 + x_2^T D_2 x_2 + x_2^T A_{22} x_2 = \mu(x_1^T x_1 + x_2^T x_2) \tag{2.2}$$

. Then the z $= \left[\begin{array}{c} D_1^{\frac{1}{2}} x_1 \\ \hline -D_2^{\frac{1}{2}} x_2 \end{array}\right]$ Then,

$$z^T \mathcal{L} z = x_1^T D_1 x_1 - x_1^T A_{11} x_1 + 2x_1^T A_{12} x_2 + x_2^T D_2 x_2 - x_2^T A_{22} x_2 \tag{2.3}$$

.

We find that $\frac{z^T \mathcal{L} z}{z^T z} = 2 - \mu \frac{x^T x}{z^T z} \geq 2 - \frac{\mu}{\delta}$. Finally using the fact that $\lambda = max\{\frac{u^T \mathcal{L} u}{u^T u} | u \neq 0\}$, we find that,

$$\lambda \geq 2 - \frac{\mu}{\delta} \tag{2.4}$$

.

Next we consider the left hand in equality in (2.1). Now let y be an eigenvector of L corresponding to λ and partition y as $\left[\begin{array}{c} y_1 \\ \hline -y_2 \end{array}\right]$ where y_1 and y_2 are nonnegative vectors. partition D and A conformally with y as as $\left[\begin{array}{c|c} \hat{D}_1 & 0 \\ \hline 0 & \hat{D}_2 \end{array}\right]$ *and* $\left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline \hat{A}_{21} & \hat{A}_{22} \end{array}\right]$

(It should be kept in mind that partitioning for A and D here may be different than the partitioning arising from x). Since $y^Ty = \lambda y^Ty$, we have

$$y_1^Ty_1 - y_1^T\hat{D}_1^{-\frac{1}{2}}\hat{A}_{11}\hat{D}_1^{-\frac{1}{2}}y_1 + 2y_1^T\hat{D}_1^{-\frac{1}{2}}\hat{A}_{12}\hat{D}_2^{-\frac{1}{2}}y_2 + y_2^Ty_2 - y_2^T\hat{D}_2^{-\frac{1}{2}}\hat{A}_{22}\hat{D}_2^{-\frac{1}{2}}y_2 = \lambda(y_1^Ty_1 + y_2^Ty_2). \tag{2.5}$$

Now let w $= \left[\begin{array}{c} \hat{D}_1^{\frac{1}{2}}y_1 \\ \hline -\hat{D}_2^{\frac{1}{2}}y_2 \end{array}\right]$ and note that w$^TQw = y_1^Ty_1 + y_1^T\hat{D}_1^{-\frac{1}{2}}\hat{A}_{11}\hat{D}_1^{-\frac{1}{2}}y_1 - 2y_1^T\hat{D}_1^{-\frac{1}{2}}\hat{A}_{12}\hat{D}_2^{-\frac{1}{2}}y_2 + y_2^Ty_2 + y_2^T\hat{D}_2^{-\frac{1}{2}}\hat{A}_{22}\hat{D}_2^{-\frac{1}{2}}y_2$. From (2.5) we find that $w^TQw = (2-\lambda)y^Ty$.
Since $\mu = min\{\frac{u^TQu}{u^Tu}|u \neq 0\}$, we find that $\mu \leq (2-\lambda)\frac{y^Ty}{w^Tw}$. Observing that $\frac{y^Ty}{w^Tw} \leq \Delta$, we thus find that $\mu \leq (2-\lambda)$ Rearranging this yields

$$\lambda \leq 2 - \frac{\mu}{\Delta} \tag{2.6}$$

□

Evidently if G is regular (so that $\delta = \Delta$) or bipartite (so that $\mu = 0$), then equality holds throughout (2.1). The next two examples show that for non-bipartite, non-regular graphs, equality can still hold in either of the inequalities in (2.1). Throughout the thesis $\mathbf{1}_p$ is used to denote an all ones vector of order p unless stated explicitly; the subscript will be suppressed when the order is clear from the context.

Example 2.1.1. *Suppose that $m \in N$ with $m \geq 2$, and let G be the graph given by $(K_m \bigcup K_m) \bigvee K_1$, where $\bigcup$ denotes the union and $\bigvee$ denotes the join operation. It should be noted that G is not bipartite, not regular, and has minimum degree m. The signless Laplacian matrix for G can be written as*

$$Q = \left[\begin{array}{c|c|c} (m-1)I+J & 0 & \mathbf{1} \\ \hline 0 & (m-1)I+J & \mathbf{1} \\ \hline \mathbf{1}^T & \mathbf{1}^T & 2m \end{array}\right];$$

Here$J$ denotes all ones matrix. From this we find that the signless Laplacian spectrum of $2m-1$, $\dfrac{4m-1 \pm \sqrt{8m+1}}{2}$, and $m-1$, the later with multiplicity

$2m-2$. *Similarly, the normalised Laplacian matrix for* G *is*

$$\mathcal{L} = \left[\begin{array}{c|c|c} \frac{m-1}{m}I - \frac{1}{m}J & 0 & -\frac{1}{m\sqrt{2}}\boldsymbol{1} \\ \hline 0 & \frac{m-1}{m}I - \frac{1}{m}J & -\frac{1}{m\sqrt{2}}\boldsymbol{1} \\ \hline -\frac{1}{m\sqrt{2}}\boldsymbol{1}^T & -\frac{1}{m\sqrt{2}}\boldsymbol{1}^T & 1 \end{array}\right];$$

It is then found that the normalised Laplacian spectrum is given by $0, \frac{1}{m}$ *and* $\frac{m+1}{m}$, *the latter with multiplicity* $2m-1$.

consequently we find that the largest normalised Laplacian eigenvalue is $\lambda = \frac{m+1}{m}$, *the smallest normalised Laplacian eigenvalue is* $\mu = m-1$, *and that equality holds in (2.4).*

Example 2.1.2. *Suppose that* $m \in N$ *with* $m \geq 3$, *and consider the graph* $Gon2m+1$ *vertices constructed as follows: start with* $K_{m,m}$, *and delete an edge from it , say between vertices* i *and* j; *then take an isolated vertex* k, *and add the edges* $k \sim i$ *and* $k \sim j$. *Observe that* G *is not regular, is not bipartite, and has maximum degree* m.

The Signless Laplacian matrix G *can be written as:*

$$Q = \left[\begin{array}{c|c|c|c|c} m & 0^T & 0 & \boldsymbol{1}^T & 1 \\ \hline 0 & mI & \boldsymbol{1} & J & 0 \\ \hline 0 & \boldsymbol{1}^T & m & 0^T & 1 \\ \hline \boldsymbol{1} & J & 0 & mI & 0 \\ \hline 1 & 0^T & 1 & 0^T & 2 \end{array}\right].$$

It now follows that the signless Laplacian spectrum of G *consists of* m *(with multiplicity* $2m-4$*),* $\frac{m+1\pm\sqrt{m^2+2m-3}}{2}$, *and the roots of the cubic* $z^3-(3m+1)z^2+(2m^2+4m-3)z-(4m^2-8m+4)$. *Computations shows that the smallest signless Laplacian eigenvalue for* G *is* $\mu = \frac{m+1-\sqrt{m^2+2m-3}}{2}$.

The normalised Laplacian matrix for G is given by

$$\mathcal{L} = \left[\begin{array}{c|c|c|c|c} 1 & 0^T & 0 & -\frac{1}{m}\mathbf{1}^T & -\frac{1}{\sqrt{2m}} \\ \hline 0 & I & -\frac{1}{m}\mathbf{1} & -\frac{1}{m}J & 0 \\ \hline 0 & -\frac{1}{m}\mathbf{1}^T & 1 & 0^T & -\frac{1}{\sqrt{2m}} \\ \hline -\frac{1}{m}\mathbf{1} & -\frac{1}{m}J & 0 & I & 0 \\ \hline -\frac{1}{\sqrt{2m}} & 0^T & -\frac{1}{\sqrt{2m}} & 0^T & 1 \end{array}\right].$$

The eigenvalues of $\mathcal{L}$ are 0,1 (with multiplicity $2m-4$) ,$2 - \frac{1}{m}(\frac{m+1\pm\sqrt{m^2+2m-3}}{2})$, and $\frac{2+\frac{1}{m}\pm\sqrt{\frac{4}{m}-\frac{3}{m^2}}}{2}$. It follows that the largest normalised Laplacian eigenvalue is $\lambda = 2 - \frac{1}{m}(\frac{m+1\pm\sqrt{m^2+2m-3}}{2})$, and that equality hold in (2.6).

2.2 Small Rayleigh quotients for the signless Laplacian matrix

From the results of section 2.2, we find that both the largest normalised Laplacian eigenvalue, and the smallest signless Laplacian eigenvalue can be thought of as providing a measure of how close a connected graph is to being bipartite. In this section, we focus on the signless Laplacian matrix, as the analysis for that matrix is somewhat more tractable than for the normalised Laplacian matrix.

Recall that the algebraic connectivity of a graph is the second smallest eigenvalue of its Laplacian matrix; see ((Abreu (2007))) and ((Kirkland (2007))) for surveys on this remarkable quantity. The algebraic connectivity plays a role in the following result, which provides a sufficient condition for a sub-graph to be bipartite.

Theorem 2.2.1. *Let G be a graph on k vertices with signless Laplacian matrix Q, and let $x \in \mathbb{R}^k$ be a vector with at least one positive entry and at least one negative entry. By permuting the entries of x and simultaneously permuting the rows and columns of Q, we assume without loss of generality that $x_j < 0, j = 1, ..., l, x_j > 0, j = l+1, ..., n$, and $x_j = 0, j = n+1, ..., k$. Let y denote the subvector of x on its first n entries , let $s = \left[\begin{array}{c} \mathbf{1}_l \\ \hline -\mathbf{1}_{n-l} \end{array}\right]$, let $z = y - \frac{y^T s}{n} s$, and let H_0 denote the* *subgraph of G induced by the edges in G of the form i where $1 \leq i \leq l < j \leq n$ Set $\nu = \frac{x^TQx}{x^Tx}$ and $\theta = \frac{n^2y^Ty}{(y^Ts)^2} - n$; denote the algebraic connectivity of H_0 by α, and let $\epsilon = min\{(y_i + y_j)^2 | i, j = 1, ..., l, i \neq j\}\{(y_i + y_j)^2 | i, j = 1, ..., l, i \neq j\}$. If*

$$\nu < \frac{\frac{n^2\epsilon}{(y^Ts)^2} + \alpha\theta}{n + \theta}, \tag{2.7}$$

.

then the nonzero entries of x induce a bipartite subgraph of G.

Proof. We begin by remarking that θ provides a measure of how close y is to s, since $\theta = 0$ if and only if $y = s$, by the Cauchy-Shwarz inequality (see A.5). Observing that z is orthogonal to s, we find readily that $z^Tz = \frac{(y^Ts)^2}{n^2}\theta$. We partition z conformally with s as $z = \begin{bmatrix} u \\ \hline -v \end{bmatrix}$ and let $\tilde{z} = \begin{bmatrix} u \\ \hline v \end{bmatrix}$; observe that $\tilde{z}\mathbf{1} = 0$. For each $j = 1, ..., n$, let $d_0(j) = |\{r | j \sim r, n+1 \leq r \leq k\}|$. Since $x^TQx =^T x$, we find that

$$\nu y^Ty = \sum_{i \sim j, 1 \leq i,j \leq l} (y_i + y_j)^2 + \sum_{i \sim j, l+1 \leq i,j \leq n} (y_i + y_j)^2 + \sum_{i \sim j, 1 \leq i \leq l, l+1 \leq j \leq n} (y_i + y_j)^2 + \sum_{j=1}^{n} d_0(j) y_j^2.$$

$$\begin{aligned} \nu y^Ty &\geq \sum_{i \sim j, 1 \leq i,j \leq l} (y_i + y_j)^2 + \sum_{i \sim j, l+1 \leq i,j \leq n} (y_i + y_j)^2 + \sum_{i \sim j, 1 \leq i \leq l, l+1 \leq j \leq n} (y_i + y_j)^2 \\ &= \sum_{i \sim j, 1 \leq i,j \leq l} (y_i + y_j)^2 + \sum_{i \sim j, l+1 \leq i,j \leq n} (y_i + y_j)^2 + \tilde{z}^T L(H_0) \tilde{z}, \end{aligned}$$

Where $L(H_0)$ is the Laplacian matrix of H_0. As $z\mathbf{1} = 0$, we have $\tilde{z}^T L(H_0)\tilde{z} \geq \alpha \tilde{z}^T \tilde{z} = \alpha z^T z$.

We proceed by contraposition, so suppose that the induced subgraph on vertices $1, ..., n$, which we denote by H, is not bipartite. Then without loss of generality we may assume that $l \geq 2$ and that H contains the edge $1 \sim 2$. From the above, we have $\nu y^Ty \geq (y_1 + y_2)^2 = \alpha z^Tz \geq \epsilon + \alpha z^Tz$. Since $z^Tz = \frac{(y^Ts)^2}{n^2}\theta$ and $y^Ty = \frac{(y^Ts)^2}{n} + z^Tz$, we find that

$$\nu \geq \frac{\epsilon + \alpha\theta\frac{(y^Ts)^2}{n^2}}{(n+\theta)\frac{(y^Ts)^2}{n^2}}.$$

The conclusion now follows $\square$

Example 2.2.1. *In this example we illustrate the fact that , in the context of Theorem (2.2.1) , some constraint on ν is needed in order to conclude that the subgraph is bipartite.*

Here we consider K_3, whose corresponding signless Laplacian matrix is $Q = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. *Let* $x = \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \\ -\frac{3}{4} \end{bmatrix}$. *We find readily that* $\nu = \frac{x^TQx}{x^Tx} = 1, y^Ts = 3 = n, \theta = \frac{3}{8}$ *and* $\epsilon = \frac{9}{4}$; *the subgraph H_0 induced by the edges $1 \sim 2, 2 \sim 3$, is $K_{1,2}$, so $\alpha = 1$. All of this yields $\frac{\epsilon+\alpha\theta}{n+\theta} = \frac{7}{9}$, which is of course less than ν, as K_3 is not bipartite.*

Remark 2.2.1. *Suppose that in the context of (2.2.1), the subgraph of G on vertices $1, ..., n - H$ say - is bipartite. Let R denote the principal submatrix of Q on rows and columns $1, ..., n$, we find readily that s^TRs is the number of edges between vertices in H and vertices in $G \backslash H$. It is not difficult to determine that $s^TRs \geq \mu n$, where μ denote the smallest eigenvalue of Q.*

It is natural to anticipate that if ν is small, then the number of edges between H and $G \backslash H$ will also be small; the results below are an attempt to reinforce that intuition.

In the next result, we continue with the notation of (2.2.1)

Theorem 2.2.2. *Suppose that $\nu < \alpha$, and let $q = |\{j | d_0(j) \geq 1, j = 1, ..., n\}|$. Then $q \leq \nu n(\frac{1+\alpha-\nu}{\alpha-\nu})$*

Proof. Let H denote the subgraph of G induced by vertices $1, ..., n$. We begin by remarking that q denotes the number of vertices in the subgraph H that are adjacent to at least one vertex in $G \backslash H$. Suppose for concreteness that $d_0(j_p) \geq 1$ for $p = 1, ..., q$. Arguing as in the proof of (2.2.1), we have $\nu n - (\alpha - \nu)z^Tz \geq \sum_{p=1}^{q} d_0(j_p)(1 + z_{j_p})^2 \geq \sum_{p=1}^{q} (1 + z_{j_p})^2$.

Set $w^T = [z_{j_1}...z_{j_q}]$. Then $\nu n - (\alpha - \nu)w^T w \geq \sum_{p=1}^{q}(1+w_i)^2$. Applying Cauchy-Schwarz inequality (twice), and letting $\beta = w^T\mathbf{1}$, we find that $\nu n - (\alpha-\nu)\frac{\beta^2}{q} \geq \frac{(q+\beta)^2}{q}$. Consequently, we find that $(1+\alpha-\nu)\beta^2 + 2q\beta + q^2 - \nu qn \leq 0$, and by considering the left hand side as a quadratic in β, and then minimising that quadratic we find that $-\frac{q^2}{1+\alpha-\nu} + q^2 - \nu qn \leq 0$. It now follows that $q \leq \nu n(\frac{1+\alpha-\nu}{\alpha-\nu}$, as desired. ☐

Remark 2.2.2. *In the context of Theorem (2.2.2) , we always (trivially) have $q \leq n$, so Theorem (2.2.2) only yields useful information in the case that $\frac{\nu(1+\alpha-\nu)}{\alpha-\nu} < 1$. It is straightforward to determine that this last holds only when either $\nu < \frac{2+\alpha-\sqrt{4+\alpha^2}}{2}$ or $\nu > \frac{2+\alpha+\sqrt{4+\alpha^2}}{2}$. Evidently the latter condition violates the hypothesis that $\nu < \alpha$ in Theorem (2.2.2), so we conclude that Theorem (2.2.2) yields a nontrivial bound on q when $\nu < \frac{2+\alpha-\sqrt{4+\alpha^2}}{2}$.*

As above, we continue with the notation of Theorem (2.2.1). In the special case that the vector x of Theorem 2.2.1 is an eigenvector corresponding to the smallest signless Laplacian eigenvalue, we can derive an upper bound on the number of edges between H and $G\backslash H$.

Theorem 2.2.3. *Let G be a graph on k vertices,and x be an eigenvector corresponding to the smallest eigenvalue μ of Q.Suppose that $x_j < 0, j = 1, ..., l, x_j > 0, j = l+1, ..., n$, and $x_j = 0, j = n+1, ..., k$. Let y denote the subvector of x on its first n entries, let $s = \left[\begin{array}{c} \mathbf{1}_l \\ \hline -\mathbf{1}_{n-l} \end{array}\right]$. Let $\theta = \frac{n^2 y^T y}{(y^T s)^2} - n$, let R denote the principal submatrix of Q on vertices $1, ..., n$ and denote the largest eigenvalue of R by ρ.*

Then $s^T Rs \leq \mu\frac{n^2}{n+\theta} + \rho\frac{n\theta}{n+\theta}$.

Proof. Note that y is an eigenvalue of R corresponding to its smallest eigenvalue, which is necessarily μ. Denote the remaining eigenvalues of R by $\lambda_2 < ... < \lambda_n \equiv \rho$, and let $u_j, j = 2, ..., n$ denote the corresponding eigenvectors, which we take (without loss of generality) to be pairwise orthogonal, and orthogonal to y. We find that $s = \frac{y^T y}{y^T y}y + \sum^{n} \frac{u_j^T s}{u_j^T u_j}u_j$.. In particular, we find that $n = s^T s = \frac{(y^T s)^2}{y^T y} + \sum_{j=2}^{n} \frac{(u_j^T s)^2}{u_j^T u_j}$.

Next we note that $s^TRs = \mu\frac{(y^Ts)^2)}{y^Ty} + \sum_{j=2}^{n}\lambda_j\frac{(u_j^Ts)^2}{u_j^Tu_j} \leq \mu\frac{(y^Ts)^2)}{y^Ty} + \rho\sum_{j=2}^{n}\frac{(u_j^Ts)^2}{u_j^Tu_j} = \mu\frac{(y^Ts)^2)}{y^Ty} + \rho(n - \frac{(y^Ts)^2)}{y^Ty})$. Using the fact that $\theta = \frac{n^2y^Ty}{(y^Ts)^2)} - n$, and rewriting inequality slightly now yields the desired conclusion. $\square$

Example 2.2.2. *In this example, we revisit the graph G of Example 2.1.2, and use it to illustrate the results of Theorems 2.2.1,2.2.2, 2.2.3 . As we saw earlier, the signless Laplacian matrix for G can be written as*

$$Q = \left[\begin{array}{c|c|c|c|c} m & 0^T & 0 & \boldsymbol{1}^T & 1 \\ \hline 0 & mI & \boldsymbol{1} & J & 0 \\ \hline 0 & \boldsymbol{1}^T & m & 0^T & 1 \\ \hline \boldsymbol{1} & J & 0 & mI & 0 \\ \hline 1 & 0^T & 1 & 0^T & 2 \end{array}\right].$$

and the smallest signless Laplacian eigenvalue is $\mu = \frac{m+1-\sqrt{m^2+2m-3}}{2}$ It is straightforward to determine that the vector $x = \left[\begin{array}{c} -\mu \\ \hline \boldsymbol{1}_{m-1} \\ \hline -1+\mu \\ \hline -\boldsymbol{1}_{m-1} \\ \hline 0 \end{array}\right]$ serves as μ-eigenvector.Applying Theorem 2.2.1 with the vector x, we note that the subgraph H identified by the nonzero entries of x is the subgraph of G formed by deleting the vertex of degree 2. Computing the relevant quantities, we find that $\nu = \mu, \epsilon = (2-\mu)^2), y^Ts = 2(m-\mu), n = 2m, \alpha = \frac{3m-2-\sqrt{m^2+4m-2}}{2}$ and $\theta = \frac{2m(m-1)\mu^2}{(m-\mu)^2}$.t is straightforward to show that as $m \to \infty, \mu$ is asymptotic to $\frac{1}{m}$; on the other hand , the right-hand side of (2.6) is asymptotic to $\frac{2}{m}$ as $m \to \infty$, so that for all sufficiently large values of m, we find that (2.6) is satisfied.

Referring to Theorem 2.2.2, we find that for the subgraph H of G, the value of q is 2. Since α is readily seen to be asymptotic to $m-2$ as $m \to \infty$ it follows that as $m \to \infty$, the expression $\nu n(\frac{1+\alpha-\nu}{\alpha-\nu})$ coverges to 2 as $m \to \infty$. Thus we find that for all sufficiently large values of m, the upper bound on q furnishes by the Theorem 2.2.2 is accurate.

Turning to Theorem 2.2.3, we find that for the subgraph H, we have $s^T Rs =$ 2.Note that the largest eigenvalue of R is readily seen to lie between $2m - 1$ and $2m$. It now follows that the expression $\mu\frac{n^2}{n+\theta} + \rho\frac{n\theta}{n+\theta}$ converges to 2 as $m \to \infty$, so that Theorem 2.2.3 provides an accurate estimate of $s^T Rs$ for this example.

Chapter 3

Computation, Results and Discussion

3.1 Inequality 2.1 for large graphs

Using NetworkX (see B.1) 100 multigraphs are radomly generated, each on 1000 vertices, whose degree sequences followed a power law distribution with exponent 2.1.(so a scale free network (see B.2)) Loops and multiple edges were then removed to generate simple loop-free graphs whose degree sequences were close to following a power law distribution. For each such graph, the connected component with the largest number of vertices is considered, and for that component, the largest normalised Laplacian eigenvalue , as well as the expressions $2 - \frac{\mu}{\Delta}$ and $2 - \frac{\mu}{\delta}$ are computed , where is the smallest signless Laplacian eigenvalue, and where δ, Δ represent the minimum and maximum degrees, respectively. The results are depicted in Figure 3.1 below, which plots $2-\frac{\mu}{\delta}$ (blue), λ (red), and $2-\frac{\mu}{\Delta}$(yellow), for each graph; here the graphs were sorted according to increasing values of . The values of for these examples ranged between approximately 0.05055 and 0.23191 (see Table C.1). Since the graphs that were generated have degree sequences that are roughly distributed according to a power law, their maximum degrees are typically quite large, while the minimum degrees are quite small. These observations are reflected in Figure 3.1: the of the blue graph corresponding to $2 - \frac{\mu}{\Delta}$ is approximately constant, while the yellow graph corresponding to $2 - \frac{\mu}{\delta}$ is more sensitive to the value of μ.

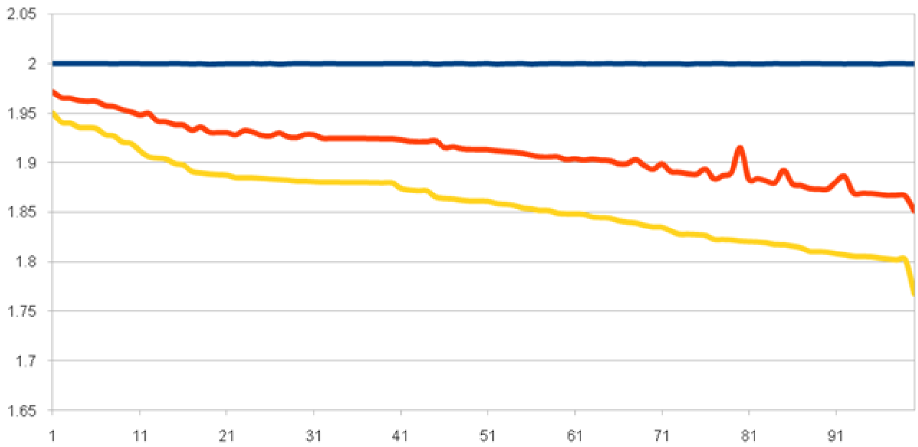

Figure 3.1: $2 - \frac{\mu}{\delta} \leq \lambda \leq 2 - \frac{\mu}{\Delta}$

Note that for all of the graphs, the quantities $2 - \frac{\mu}{\Delta}$ and $2 - \frac{\mu}{\delta}$ provide a fairly small interval in which is contained.

3.2 Identification of bipartite subgraphs in Scale Free Networks

Theorem 2.2.1 suggests a strategy for identifying bipartite subgraphs of a given graph G. The approach is as follows:

1. compute a unit eigenvector v corresponding to the smallest signless Laplacian eigenvalue for G;

2. construct the vector x from v by setting its entries of small absolute value equal to zero.

3. if it happens that (2.7) holds, then the non-zero entries of x induce a bipartite subgraph of G.

This approach is implemented on a collection of randomly generated graphs. Specifically, using NetworkX (see B.1) 500 multigraphs, each on 600 vertices are randomly generated, whose degree sequences followed a power law distribution

with exponent 2.1. Loops and multiple edges were then removed to generate simple loop-free graphs whose degree sequences were close to following a power law distribution. For each graph so generated, the connected component G containing the maximum number of vertices is considered and computed the unit eigenvector v_G for the smallest eigenvalue of its signless Laplacian matrix.
Next, the vector x_G from v_G is formed by rounding all of its entries to the first decimal place. Then the subgraph H_G induced by the nonzero entries of x_G is considered . It should be noted here that the decision to round the entries of v_G to the first decimal place is motivated by a need to introduce some zeros into the vectors with which the work is done , since an actual value of 0 is a rare occurrence when computing eigenvectors numerically. Evidently different strategies for approximating v_G will yield different subgraphs; however the results below suggest that our rounding method is reasonably successful in identifying bipartite subgraphs that are weakly connected to the rest of the graph G.
For a total of 403 different graphs G, the subgraphs H_G were connected, and the corresponding vectors x_G satisfied (2.7), so that bipartiteness was assured by Theorem 2.2.1. These bipartite subgraphs were all of small order, containing between 3 and 10 vertices, while the original connected graphs had orders ranging from 548 to 598 vertices. For each of these 403 graphs the number of edges between H_G and the rest of G was at most 9; for 208 of these graphs, there was just one edge between H_G and the rest of G.
For a further 63 graphs G, the subgraphs H_G were connected and bipartite, but (2.7) did not hold. Again, the H_Gs were of small order between 3 and 16 vertices while the orders of the connected graphs G ranged from 568 to 592. In each of these cases, there were at most 11 edges connecting H_G to the rest of G. For the remaining 34 graphs, the subgraph H_G was either not connected, or not bipartite, or contained just two vertices.
Based on these computations, it appears that there is some utility in the approach to identifying bipartite subgraphs suggested by Theorem 2.2.1.

3.3 A protein-protein interaction network

In (Bu *et al.* (2003)), the authors consider protein-protein interaction networks and are interested in identifying bipartite (or nearly bipartite) subgraphs within such networks; for finding such subgraphs leads to an enhanced understanding of the function of the corresponding proteins. The approach to identifying such bipartite subgraphs proposed in (Bu *et al.* (2003)) is to consider the network as a graph, and then use eigenvectors of the adjacency matrix corresponding to small eigenvalues in order to identify bipartite subgraphs. For the example of a protein-protein interaction network on 2617 vertices (corresponding to budding yeast), it is reported in (Bu *et al.* (2003)) that six so-called Quasi-bipartite subgraphs are identified by this adjacency matrix eigenvector method.

In view of the results in section 2.2, it is natural to conjecture that eigenvectors of the signless Laplacian matrix corresponding to small eigenvalues may also yield an effective technique for identifying bipartite subgraphs of a protein-protein interaction network. Below we describe the results of an implementation of this idea.

The dataset describing a protein-protein interaction network for budding yeast is used. That network has 2361 vertices, and 7182 edges, of which 536 are loops. 13 loops is been removed to yield a loop-free graph G on 2361 vertices with 6646 edges. The graph G consists of 101 connected components. One hundred of these connected components are either trees (and so, necessarily bipartite) or isolated vertices, of orders ranging from one to eight vertices; taken together they contain 137 vertices and 37 edges. The remaining connected component of G, which is denoted by $\hat{G}$, is a graph on 2224 vertices, with 6609 edges. Since the smallest signless Laplacian eigenvalue for $\hat{G}$ is positive (approximately 0.0609), $\hat{G}$ is not bipartite. Figure 3.2 gives a depiction of $\hat{G}$.

Unit eigenvectors of the signless Laplacian matrix for $\hat{G}$ corresponding to its 50 smallest eigenvalues are considered. For each such unit eigenvector v, the vector x_v is formed by rounding the entries of v to the first decimal place. Then the subgraph of $\hat{G}$, say H_v , induced by the nonzero entries of x_v is considered.

For a total of 12 such eigenvectors v, the corresponding subgraph H_v was connected and the vector x_v satisfied (2.7), thus ensuring that H_v was bipartite. These subgraphs were of small order, between 3 and 8 vertices, and in each case were joined to the rest of G by at most 15 edges. For 7 of these H_v s, there was just one edge joining H_v to the rest of $\hat{G}$. For a further 15 eigenvectors v, the corresponding subgraph H_v was connected and bipartite, but (2.7) was not satisfied by x_v . Again the subgraphs were of orders between 3 and 8, and were

joined to the rest of $\hat{G}$ by at most 12 edges. For the remaining 23 eigenvectors under consideration, the corresponding subgraph was either not connected, or not bipartite, or consisted of just two vertices.
Based on these results, it seems that the technique of using signless Laplacian eigenvectors for small eigenvalues to identify bipartite subgraphs represents an improvement on the adjacency eigenvector approach employed in (Bu *et al.* (2003)).

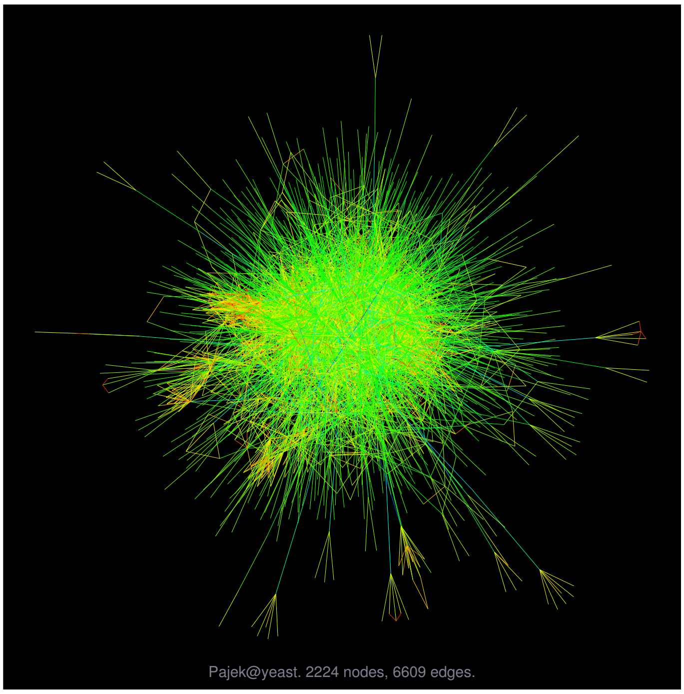

Figure 3.2: The connected component $\hat{G}$

Chapter 4

Conclusion & Scope for Further Research

In this thesis, an attempt is made to explore the possible real world application of the emerging Q-theory or the spectral graph theory based on the signless Laplacian matrix. The signless Laplacian matrix has its special importance in studying bipartite graphs and bipartite graphs, in turn, have its own importance in modelling different complex systems. The uniqueness of this work lies in exploring the technique to identify bipartite subgraphs from the entries of eigenvector corresponding to an eigenvalue of the Q-spectrum and it is described in Chapter 2. In Chapter 2, we come across four theorems in total. Theorem 2.1.1 gives the the relation between the two extreme eigenvalues i.e, μ-the smallest eigenvalue of the signless Laplacian spectrum and λ-the largest eigenvalue of the normalized Laplacian spectrum. Theorem 2.2.1 gives the sufficient condition for the subgraphs guaranteed to be bipartite. Theorem 2.2.2 and Theorem 2.2.3 gives the characteristic of those bipartite subgraphs. These results are then applied to a protein protein interaction network and it is shown that number of bipartite components identified through this method is more than that of the adjacency eigenvector method.

Coming to the future scope of this research, it can be said that many conjectures have been stated within this theory regarding eigenvalues of the signless Laplacian matrix(Cvetkovic *et al.* (2007)), some of them have been proved or disproved and many of them are still in research. Whatever researches have been made so far is to enrich the Q-theory itself but they lack of exploring interdisciplinary applications. From that point of view, this work is the first attempt. In this work, graphs are taken as undirected and simple. But many of the complex networks are not fully undirected and also simple! like the metabolic network. Therefore further generalization of this theory for digraphs and multigraphs can be made so that it can address the reality in a more unconstrained way.

Appendix A

Appdx A

A.1 The Perron-Frobenius Theorem of Non-negative Matrices

A matrix or a vector can be said positive or negative according to the entries. If the elements are positive then the matrix or the vector is said to be a positive matrix or vector and if non-negative then it is said non-negative matrix or vector. A non-negative matrix of dimension nxn is said to be irreducible if there is no permutation of co-ordinates such that

$$P^{'}AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

*Where P is an nxn permutation matrix (each row and each column have exactly one 1 entry and all others 0), A_{11} is rxr, and A_{22} is $(n-r)x(n-r)$. The **Perron-Frobenius** theorem of irreducible matrix states the following:*

Theorem A.1.1. *If A is nxn, non-negative, irreducible, then,*

- *one of its eigenvalues(called the dominant) is positive and greater than or equal to(in absulate value) all other eigenvalues.*
- *there is a positive eigenvector corresponding to that eigenvalue and*
- *that eigenvalue is a simple root of the charateristic equation of A.*

.

A matrix of dimension nxn is said to be primitive if and only if $A^k > 0$ for some power k. The next theorem due to ***Perron-Frobenius*** *is based on this.*

Theorem A.1.2. *If A is nxn, non-negative,primitive,then,*

- *one of its eigenvalues(called the dominant) is positive and greater than or equal to(in absulate value) all other eigenvalues.*
- *there is a positive eigenvector corresponding to that eigenvalue and*
- *that eigenvalue is a simple root of the charateristic equation of A.*

.

Applications of this theorem is huge especially in algebraic graph theory (adjacency matrix of a strongly connected graph is irreducible and hence the theorem is applicable here), population dynamics model , finite Markov chains and many discrete models.

A.2 Spectral Uncertainty

*The term **spectral uncertainty** is related to cospectrality and isomorphism of graphs. So we should know what are cospectral graphs and how it is related to graph isomorphism.*

Graphs with the same spectrum of an associated matrix M *(this matrix may be* A *or* L *or* Q*) are called cospectral graphs with respect to* M *, or* M *cospectral graphs. Let* H *be a graph copectral with* G *but not isomorphic to* G*, is called cospectral mate of* G*. Now the spectral uncertainty with respect to* M *is defined as the ratio:* $|\mathcal{G}'|/|\mathcal{G}|$ *,where* $\mathcal{G}'$ *is the cospectral mate of* $\mathcal{G}$ *with respect to* M *and both the graphs are finite.*

A.3 Line graph of a graph

Given a graph G*, its line graph denoted by* $L(G)$ *is graph such that*

- *each vertex of* $L(G)$ *represents an edge of* G*; and*
- *two vertices of* $L(G)$ *are adjacent if and only if their corresponding edges share a common endpoint in* G

.

A.4 Rayleigh quotient

*The **Rayleigh quotient** denoted by* $R(A, x)$ *for a given **Hermitian matrix*** A *and non-zero vector* x *is defined as*

$$\frac{x^*Ax}{x^*x}.$$

For real matrices and vectors , the condition of being Hermitian reduces to that of being symmetric, and the conjugate transpose x^* *to the usual transpose* x^T*.* $R(A, cx) = R(A, x)$ *,where* c *is a real scaler.* $\lambda_{min} \leq R(A, x) \leq \lambda_{max}$ *,where* λ *is the eigenvalue of* A*.*

A.5 Cauchy-Schwarz inequality

The ***Cauchy-Schwarz inequality*** *states that for all vectors* x *and* y *of an inner product space,*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle.\langle y, y\rangle$$

where $\langle ., .\rangle$ *is the inner product.*

Appendix B

Appdx B

B.1 NetworkX

NetworkX is a Python package for the creation, manipulation, and analysis of complex networks. It consists of data structures for graphs (or networks) along with graph algorithms, generators, and drawing tools. The structure of NetworkX can be seen by the organization of its source code. The package provides classes for graph objects, generators to create standard graphs, IO routines for reading in existing datasets, algorithms to analyse the resulting networks and some basic drawing tool. Most of the NetworkX API is provided by functions which take a graph object as an argument. Methods of the graph object are limited to basic manipulation and reporting. This provides modularity of code and documentation.

B.2 Scale Free Networks

Scale free *networks are those networks whose degree distribution follow power law with the exponent value between 2 and 3 i.e,* $P(k) \sim ck^{-\gamma}$*, where* $2 < \gamma < 3$*. The term scale free means maintaining self similarity in spite of adding more nodes to the network. The properties of a scale free network are given below:*

- *Scale-free networks are more robust against failure. By this we mean that the network is more likely to stay connected than a random network after the removal of randomly chosen nodes.*
- *Scale-free networks are more vulnerable against non-random attacks. This means that the network quickly disintegrates when nodes are removed according to their degree.*
- *Scale-free networks have short average path lengths* $L \sim \log N / \log \log k$*., where* N *is the number of nodes in the network and* k *is the scaling index*

. The exmaples of such networks are: co-author scientific networks, the internet and world-wide web, and protein-protein interaction and gene regulatory networks.

Appendix C

Appdx C

Table C.1: Validation of the inequality 2.1. [1]

μ	λ	$U.B.or 2-\frac{\mu}{\Delta}$	$L.B.or 2-\frac{\mu}{\delta}$	$U.B.$-λ	λ-$L.B.$
0.050554793	1.971044746	1.999877887	1.949445207	0.02883314	0.02159954
0.059689155	1.965355226	1.999872186	1.940310845	0.03451696	0.02504438
0.060481853	1.964842544	1.99981275	1.939518147	0.03497021	0.0253244
0.064697202	1.962370738	1.999845959	1.935302798	0.03747522	0.02706794
0.06506431	1.961777463	1.999784555	1.93493569	0.03800709	0.02684177
0.065836439	1.961618205	1.999920965	1.934163561	0.03830276	0.02745464
0.072037663	1.957488116	1.999896498	1.927962337	0.04240838	0.02952578
0.073430067	1.956476204	1.999507181	1.926569933	0.04303098	0.02990627
0.079467746	1.953232465	1.999846291	1.920532254	0.04661383	0.03270021
0.080949844	1.951102034	1.999866859	1.919050156	0.04876482	0.03205188

[1] Unless specified , the symbols in the table bears the same meaning as that in the previous chapters.

0.0882469	1.947888956	1.999730955	1.9117531	0.051842	0.03613586
0.094276307	1.949389807	1.999628833	1.905723693	0.05023903	0.04366611
0.09561938	1.941916544	1.99973065	1.90438062	0.05781411	0.03753592
0.096624907	1.941129998	1.999624028	1.903375093	0.05849403	0.0377549
0.101149647	1.93816801	1.99984948	1.898850353	0.06168147	0.03931766
0.102621575	1.937713684	1.999661315	1.897378425	0.06194763	0.04033526
0.109152295	1.932706906	1.999309163	1.890847705	0.06660226	0.0418592
0.1105997	1.935842117	1.999559364	1.8894003	0.06371725	0.04644182
0.111740876	1.930540056	1.998925569	1.888259124	0.06838551	0.04228093
0.112299558	1.930190242	1.999241219	1.887700442	0.06905098	0.0424898
0.112740419	1.930221529	1.99951405	1.887259581	0.06929252	0.04296195
0.11522306	1.927973768	1.999579478	1.88477694	0.07160571	0.04319683
0.115334527	1.932407121	1.999461054	1.884665473	0.06705393	0.04774165
0.115503472	1.930679474	1.999850384	1.884496528	0.06917091	0.04618295
0.116024563	1.927470918	1.999351818	1.883975437	0.0718809	0.04349548
0.116756965	1.926809107	1.99983984	1.883243035	0.07303073	0.04356607
0.117298815	1.929960417	1.998961957	1.882701185	0.06900154	0.04725923
0.117864215	1.926014981	1.999587887	1.882135785	0.07357291	0.0438792
0.118742237	1.925339538	1.9998404	1.881257763	0.07450086	0.04408178
0.118807419	1.928412445	1.999804271	1.881192581	0.07139183	0.04721986
0.119418151	1.928029327	1.999638127	1.880581849	0.0716088	0.04744748
0.119854254	1.924475659	1.999747675	1.880145746	0.07527202	0.04432991
0.119908123	1.924433402	1.999842847	1.880091877	0.07540944	0.04434152

0.119922191	1.92442784	1.999568625	1.880077809	0.07514079	0.04435003
0.120042786	1.924332998	1.999629498	1.879957214	0.0752965	0.04437578
0.120088079	1.9242982	1.999612619	1.879911921	0.07531442	0.04438628
0.120122876	1.924267783	1.999690405	1.879877124	0.07542262	0.04439066
0.120304759	1.924123829	1.999465312	1.879695241	0.07534148	0.04442859
0.120457049	1.924003483	1.999727473	1.879542951	0.07572399	0.04446053
0.120539473	1.923938753	1.999874699	1.879460527	0.07593595	0.04447823
0.12596727	1.923010347	1.999734246	1.87403273	0.0767239	0.04897762
0.127727934	1.921527175	1.999765206	1.872272066	0.07823803	0.04925511
0.12846818	1.921028124	1.999616513	1.87153182	0.07858839	0.0494963
0.128512795	1.921147374	1.999857208	1.871487205	0.07870983	0.04966017
0.134690344	1.922022579	1.998955889	1.865309656	0.07693331	0.05671292
0.136181213	1.915213246	1.999608675	1.863818787	0.08439543	0.05139446
0.136632374	1.916019755	1.999705534	1.863367626	0.08368578	0.05265213
0.137947846	1.913845262	1.99983279	1.862052154	0.08598753	0.05179311
0.138776383	1.913247834	1.999291957	1.861223617	0.08604412	0.05202422
0.138917234	1.913092681	1.999632494	1.861082766	0.08653981	0.05200992
0.139090143	1.912952026	1.999847656	1.860909857	0.08689563	0.05204217
0.141330878	1.911930282	1.999076269	1.858669122	0.08714599	0.05326116
0.14199663	1.91121046	1.99951537	1.85800337	0.08830491	0.05320709
0.143058484	1.9105477	1.999683499	1.856941516	0.0891358	0.05360618
0.145875463	1.909420249	1.99982232	1.854124537	0.09040207	0.05529571
0.146653479	1.907356977	1.999094732	1.853346521	0.09173775	0.05401046

0.14836962	1.905919006	1.999589004	1.85163038	0.09367	0.05428863
0.148453353	1.905788005	1.999723551	1.851546647	0.09393555	0.05424136
0.15092184	1.906000198	1.999777729	1.84907816	0.09377753	0.05692204
0.151632036	1.902958281	1.999590184	1.848367964	0.0966319	0.05459032
0.151644994	1.903868584	1.999711702	1.848355006	0.09584312	0.05551358
0.152121827	1.902515464	1.99984004	1.847878173	0.09732458	0.05463729
0.154745324	1.903488816	1.999422592	1.845254676	0.09593378	0.05823414
0.155358673	1.902473301	1.999724052	1.844641327	0.09725075	0.05783197
0.155855546	1.901966211	1.99976738	1.844144454	0.09780117	0.05782176
0.158503584	1.898967613	1.99956811	1.841496416	0.1006005	0.0574712
0.159961355	1.898938183	1.999793064	1.840038645	0.10085488	0.05889954
0.16086762	1.902973635	1.999764813	1.83913238	0.09679118	0.06384126
0.163228138	1.896963679	1.999399897	1.836771862	0.10243622	0.06019182
0.164899994	1.893311814	1.99971812	1.835100006	0.10640631	0.05821181
0.165132213	1.898731835	1.999604947	1.834867787	0.10087311	0.06386405
0.168688394	1.891008754	1.99955956	1.831311606	0.10855081	0.05969715
0.17205512	1.890212745	1.999589367	1.82794488	0.10937662	0.06226787
0.172078791	1.888799893	1.999033265	1.827921209	0.11023337	0.06087868
0.172509264	1.88842168	1.999772715	1.827490736	0.11135103	0.06093094
0.17333741	1.893440229	1.999548601	1.82666259	0.10610837	0.06677764
0.177263451	1.883725835	1.999813013	1.822736549	0.11608718	0.06098929
0.177485474	1.886917096	1.999819812	1.822514526	0.11290272	0.06440257
0.177804814	1.890178355	1.999265269	1.822195186	0.10908691	0.06798317

0.178977404	1.914917485	1.999735241	1.821022596	0.08481776	0.09389489
0.179441485	1.883135354	1.999545718	1.820558515	0.11641036	0.06257684
0.179764725	1.883951684	1.999201046	1.820235275	0.11524936	0.06371641
0.180666888	1.881556126	1.999498148	1.819333112	0.11794202	0.06222301
0.182571697	1.879906777	1.99979893	1.817428303	0.11989215	0.06247847
0.182655421	1.892139517	1.999681785	1.817344579	0.10754227	0.07479494
0.184079498	1.878616623	1.999455386	1.815920502	0.12083876	0.06269612
0.185734773	1.877088481	1.999806929	1.814265227	0.12271845	0.06282325
0.18947229	1.873982387	1.99973426	1.81052771	0.12575187	0.06345468
0.189614732	1.873511087	1.999810005	1.810385268	0.12629892	0.06312582
0.190090011	1.873363001	1.999507539	1.809909989	0.12614454	0.06345301
0.191633677	1.880657397	1.999630051	1.808366323	0.11897265	0.07229107
0.192838055	1.886232524	1.999389753	1.807161945	0.11315723	0.07907058
0.194268009	1.86952439	1.999667918	1.805731991	0.13014353	0.0637924
0.194664642	1.869143324	1.999655461	1.805335358	0.13051214	0.06380797
0.19473318	1.869086878	1.999625513	1.80526682	0.13053864	0.06382006
0.195964445	1.868094157	1.999129047	1.804035555	0.13103489	0.0640586
0.196960045	1.867069508	1.999779687	1.803039955	0.13271018	0.06402955
0.197763613	1.867332419	1.999734189	1.802236387	0.13240177	0.06509603
0.231909832	1.851537789	1.999653349	1.768090168	0.14811556	0.08344762
0.231909832	1.851537789	1.999653349	1.768090168	0.14811556	0.08344762

Table C.2: Computation results on $\hat{G}$s which satisfied theorem 2.2.2

$\hat{G}_n$[1]	H_b^G[2]	H_n^G[3]	s^TQs	ν	$\frac{\epsilon \frac{n^2}{y^T s}+\alpha\theta}{n+\theta}$	α	θ	ϵ
590	TRUE	5	4	0.132653	0.325259063	(0.38196601125+0j)	1.125	0.25
581	TRUE	5	4	0.174757	0.356937409	(0.38196601125+0j)	2.132963989	0.25
579	TRUE	6	2	0.089109	0.296058388	(0.38196601125+0j)	0.873345936	0.25
564	TRUE	5	2	0.155963	0.403164156	(0.38196601125+0j)	1.179138322	0.36
582	TRUE	3	3	0.307692	0.525641026	(1+0j)	0.65625	0.36
586	TRUE	5	4	0.132653	0.325259063	(0.38196601125+0j)	1.125	0.25
579	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
579	TRUE	5	3	0.122449	0.325259063	(0.38196601125+0j)	1.125	0.25
583	TRUE	5	1	0.094737	0.552925039	(0.518805695908+0j)	0.385487528	0.49
578	TRUE	4	2	0.238938	0.684306562	(0.585786437627+0j)	1.008310249	0.64
572	TRUE	4	1	0.14433	0.875815036	(0.585786437627+0j)	0.299168975	0.81
582	TRUE	4	7	0.205882	0.600995247	(0.585786437627+0j)	1.037037037	0.49
561	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
580	TRUE	6	3	0.121212	0.230633378	(0.267949192431+0j)	2.081632653	0.16
596	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
578	TRUE	4	2	0.265306	0.928146627	(0.585786437627+0j)	0.839506173	0.81
592	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
588	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64

[1]Number of nodes in Ĝ

[2]Is the subgraph H of G bipartite?

[3]Number of nodes in H^G

567	TRUE	5	2	0.141509	0.329538833	(0.38196601125+0j)	1.625	0.25
578	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
590	TRUE	3	2	0.215686	0.428899083	(1+0j)	1.037037037	0.25
565	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
570	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
580	TRUE	4	1	0.181818	0.869955872	(0.585786437627+0j)	0.387811634	0.81
576	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
586	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
580	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
585	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
584	TRUE	5	2	0.112245	0.325259063	(0.38196601125+0j)	1.125	0.25
584	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
578	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
584	TRUE	5	2	0.135417	0.205533391	(0.38196601125+0j)	1.140495868	0.16
581	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
576	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
568	TRUE	7	2	0.0625	0.203855622	(0.260322690258+0j)	1.166666667	0.16
575	TRUE	6	3	0.149533	0.168002785	(0.267949192431+0j)	2.734693878	0.09
587	TRUE	6	2	0.083333	0.312305486	(0.324869129433+0j)	1.140495868	0.25
593	TRUE	5	7	0.203883	0.356937409	(0.38196601125+0j)	2.132963989	0.25
586	TRUE	4	2	0.161905	0.691812857	(0.585786437627+0j)	0.653739612	0.64
585	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
581	TRUE	6	2	0.101852	0.21596245	(0.267949192431+0j)	2.033057851	0.16

590	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
582	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
569	TRUE	3	2	0.210526	0.412672624	(1+0j)	1.259515571	0.16
590	TRUE	5	2	0.166667	0.464840444	(0.518805695908+0j)	1.375	0.36
578	TRUE	7	2	0.156522	0.157829481	(0.198062264195+0j)	0.729766804	0.16
582	TRUE	5	5	0.184466	0.356937409	(0.38196601125+0j)	2.132963989	0.25
573	TRUE	5	1	0.094737	0.552925039	(0.518805695908+0j)	0.385487528	0.49
584	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
577	TRUE	5	2	0.138298	0.34322638	(0.518805695908+0j)	0.875	0.25
582	TRUE	6	3	0.121212	0.230633378	(0.267949192431+0j)	2.081632653	0.16
577	TRUE	5	1	0.094737	0.552925039	(0.518805695908+0j)	0.385487528	0.49
586	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
578	TRUE	5	2	0.11215	0.427603244	(0.518805695908+0j)	1.065759637	0.36
577	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
584	TRUE	6	2	0.111111	0.230368113	(0.324869129433+0j)	2.033057851	0.16
584	TRUE	6	2	0.170213	0.239129514	(0.485863070665+0j)	0.991735537	0.16
580	TRUE	5	5	0.184466	0.356937409	(0.38196601125+0j)	2.132963989	0.25
581	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
579	TRUE	5	4	0.132653	0.325259063	(0.38196601125+0j)	1.125	0.25
574	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
573	TRUE	6	2	0.122449	0.322660655	(0.38196601125+0j)	1.289256198	0.25
578	TRUE	3	1	0.207921	0.402640264	(1+0j)	0.14532872	0.36
583	TRUE	5	4	0.174757	0.356937409	(0.38196601125+0j)	2.132963989	0.25

576	TRUE	5	1	0.126316	0.543130199	(0.38196601125+0j)	0.385487528	0.49
580	TRUE	5	3	0.165049	0.356937409	(0.38196601125+0j)	2.132963989	0.25
594	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
589	TRUE	5	3	0.122449	0.325259063	(0.38196601125+0j)	1.125	0.25
580	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
579	TRUE	5	2	0.141509	0.329538833	(0.38196601125+0j)	1.625	0.25
579	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
580	TRUE	4	2	0.156863	0.600995247	(0.585786437627+0j)	1.037037037	0.49
584	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
574	TRUE	6	2	0.140187	0.168002785	(0.267949192431+0j)	2.734693878	0.09
586	TRUE	6	3	0.132653	0.322660655	(0.38196601125+0j)	1.289256198	0.25
571	TRUE	6	1	0.069307	0.288803503	(0.324869129433+0j)	0.873345936	0.25
586	TRUE	3	2	0.269231	0.525641026	(1+0j)	0.65625	0.36
583	TRUE	5	1	0.128713	0.533556089	(0.38196601125+0j)	0.725623583	0.49
590	TRUE	6	4	0.157895	0.349602834	(0.38196601125+0j)	1.755102041	0.25
573	TRUE	5	2	0.141509	0.329538833	(0.38196601125+0j)	1.625	0.25
592	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
578	TRUE	4	1	0.14433	0.875815036	(0.585786437627+0j)	0.299168975	0.81
579	TRUE	4	3	0.166667	0.600995247	(0.585786437627+0j)	1.037037037	0.49
548	TRUE	5	1	0.139785	0.714949111	(0.518805695908+0j)	0.272108844	0.64
583	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
586	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
569	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49

565	TRUE	4	2	0.265306	0.928146627	(0.585786437627+0j)	0.839506173	0.81
579	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
585	TRUE	4	2	0.161905	0.691812857	(0.585786437627+0j)	0.653739612	0.64
584	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
582	TRUE	4	1	0.181818	0.869955872	(0.585786437627+0j)	0.387811634	0.81
581	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
588	TRUE	3	2	0.210526	0.412672624	(1+0j)	1.259515571	0.16
580	TRUE	6	2	0.170213	0.320137023	(0.38196601125+0j)	0.991735537	0.25
585	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
581	TRUE	3	2	0.210526	0.412672624	(1+0j)	1.259515571	0.16
590	TRUE	3	2	0.210526	0.653939886	(1+0j)	1.259515571	0.49
581	TRUE	8	4	0.150538	0.212635256	(0.253786811873+0j)	1.5232	0.16
576	TRUE	4	2	0.161905	0.691812857	(0.585786437627+0j)	0.653739612	0.64
586	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
572	TRUE	4	3	0.166667	0.600995247	(0.585786437627+0j)	1.037037037	0.49
579	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
577	TRUE	5	4	0.174757	0.356937409	(0.38196601125+0j)	2.132963989	0.25
564	TRUE	7	1	0.09901	0.273664671	(0.225377100496+0j)	0.9184	0.25
580	TRUE	6	1	0.073684	0.406422117	(0.38196601125+0j)	0.465028355	0.36
580	TRUE	5	6	0.194175	0.356937409	(0.38196601125+0j)	2.132963989	0.25
585	TRUE	5	1	0.094737	0.552925039	(0.518805695908+0j)	0.385487528	0.49
571	TRUE	5	6	0.153061	0.325259063	(0.38196601125+0j)	1.125	0.25
579	TRUE	5	3	0.122449	0.325259063	(0.38196601125+0j)	1.125	0.25

579	TRUE	5	2	0.14433	0.418200929	(0.518805695908+0j)	0.498866213	0.36
573	TRUE	5	3	0.122449	0.325259063	(0.38196601125+0j)	1.125	0.25
585	TRUE	5	3	0.205882	0.464840444	(0.518805695908+0j)	1.375	0.36
579	TRUE	5	4	0.222222	0.504080734	(0.518805695908+0j)	1.855955679	0.36
582	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
588	TRUE	4	2	0.156863	0.600995247	(0.585786437627+0j)	1.037037037	0.49
569	TRUE	6	4	0.131313	0.230633378	(0.267949192431+0j)	2.081632653	0.16
588	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
575	TRUE	3	2	0.210526	0.412672624	(1+0j)	1.259515571	0.16
579	TRUE	5	2	0.196078	0.268762013	(0.518805695908+0j)	1.375	0.16
577	TRUE	5	5	0.141414	0.144298687	(0.38196601125+0j)	2.081632653	0.04
585	TRUE	5	1	0.142857	0.715049504	(0.38196601125+0j)	0.158730159	0.64
566	TRUE	5	2	0.112245	0.325259063	(0.38196601125+0j)	1.125	0.25
589	TRUE	4	1	0.193548	0.888289384	(0.585786437627+0j)	0.121883657	0.81
579	TRUE	4	4	0.176471	0.600995247	(0.585786437627+0j)	1.037037037	0.49
581	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
598	TRUE	5	3	0.205882	0.464840444	(0.518805695908+0j)	1.375	0.36
585	TRUE	4	2	0.161905	0.691812857	(0.585786437627+0j)	0.653739612	0.64
586	TRUE	5	8	0.213592	0.356937409	(0.38196601125+0j)	2.132963989	0.25
570	TRUE	4	4	0.176471	0.600995247	(0.585786437627+0j)	1.037037037	0.49
568	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
580	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
577	TRUE	5	5	0.184466	0.356937409	(0.38196601125+0j)	2.132963989	0.25

580	TRUE	5	1	0.094737	0.552925039	(0.518805695908+0j)	0.385487528	0.49
586	TRUE	8	2	0.131313	0.177371114	(0.198062264195+0j)	0.691358025	0.16
585	TRUE	4	7	0.205882	0.600995247	(0.585786437627+0j)	1.037037037	0.49
590	TRUE	5	6	0.194175	0.356937409	(0.38196601125+0j)	2.132963989	0.25
579	TRUE	5	1	0.142857	0.715049504	(0.38196601125+0j)	0.158730159	0.64
582	TRUE	4	3	0.166667	0.600995247	(0.585786437627+0j)	1.037037037	0.49
569	TRUE	4	8	0.215686	0.600995247	(0.585786437627+0j)	1.037037037	0.49
570	TRUE	5	3	0.122449	0.325259063	(0.38196601125+0j)	1.125	0.25
588	TRUE	4	2	0.156863	0.600995247	(0.585786437627+0j)	1.037037037	0.49
587	TRUE	6	5	0.141414	0.230633378	(0.267949192431+0j)	2.081632653	0.16
584	TRUE	4	6	0.196078	0.600995247	(0.585786437627+0j)	1.037037037	0.49
583	TRUE	5	2	0.223301	0.424061401	(0.518805695908+0j)	0.839002268	0.36
583	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
580	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
594	TRUE	4	6	0.196078	0.600995247	(0.585786437627+0j)	1.037037037	0.49
586	TRUE	6	2	0.12766	0.137018683	(0.324869129433+0j)	2.46	0.04
578	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
584	TRUE	5	1	0.126316	0.543130199	(0.38196601125+0j)	0.385487528	0.49
580	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
584	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
585	TRUE	6	2	0.122449	0.322660655	(0.38196601125+0j)	1.289256198	0.25
585	TRUE	6	4	0.2	0.217322004	(0.38196601125+0j)	3	0.09
574	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49

581	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
575	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
590	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
581	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
581	TRUE	4	2	0.161905	0.691812857	(0.585786437627+0j)	0.653739612	0.64
585	TRUE	6	2	0.150442	0.212987464	(0.324869129433+0j)	1.689981096	0.16
578	TRUE	4	3	0.157407	0.35890411	(1+0j)	2.033057851	0.04
590	TRUE	6	1	0.069307	0.281571102	(0.267949192431+0j)	0.873345936	0.25
573	TRUE	5	8	0.213592	0.356937409	(0.38196601125+0j)	2.132963989	0.25
582	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
587	TRUE	5	4	0.132653	0.325259063	(0.38196601125+0j)	1.125	0.25
590	TRUE	4	2	0.188119	0.353960396	(1+0j)	0.476454294	0.25
592	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
592	TRUE	6	2	0.09375	0.218555486	(0.324869129433+0j)	1.140495868	0.16
586	TRUE	6	1	0.117117	0.457936812	(0.267949192431+0j)	0.3936	0.49
575	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
585	TRUE	5	3	0.150943	0.329538833	(0.38196601125+0j)	1.625	0.25
582	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
590	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
585	TRUE	6	4	0.131313	0.230633378	(0.267949192431+0j)	2.081632653	0.16
588	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
587	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
583	TRUE	6	1	0.085106	0.303964424	(0.267949192431+0j)	0.991735537	0.25

581	TRUE	5	1	0.13	0.192	(1+0j)	0.165289256	0.16
578	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
590	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
590	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
585	TRUE	7	5	0.151786	0.165711336	(0.321719649958+0j)	2.527777778	0.09
578	TRUE	4	1	0.181818	0.869955872	(0.585786437627+0j)	0.387811634	0.81
582	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
578	TRUE	5	4	0.132653	0.325259063	(0.38196601125+0j)	1.125	0.25
573	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
589	TRUE	6	4	0.17757	0.185823512	(0.324869129433+0j)	2.734693878	0.09
588	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
573	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
585	TRUE	4	2	0.188119	0.353960396	(1+0j)	0.476454294	0.25
574	TRUE	5	5	0.138889	0.179349238	(0.518805695908+0j)	2.033057851	0.04
588	TRUE	5	1	0.103774	0.50729257	(0.518805695908+0j)	0.475206612	0.49
580	TRUE	4	1	0.181818	0.869955872	(0.585786437627+0j)	0.387811634	0.81
579	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
580	TRUE	4	2	0.156863	0.600995247	(0.585786437627+0j)	1.037037037	0.49
583	TRUE	6	2	0.14433	0.171488913	(0.324869129433+0j)	1.918367347	0.09
584	TRUE	7	1	0.052083	0.194961276	(0.198062264195+0j)	1.166666667	0.16
568	TRUE	4	1	0.181818	0.869955872	(0.585786437627+0j)	0.387811634	0.81
587	TRUE	3	1	0.207921	0.402640264	(1+0j)	0.14532872	0.36
585	TRUE	5	5	0.184466	0.356937409	(0.38196601125+0j)	2.132963989	0.25

582	TRUE	4	2	0.161905	0.691812857	(0.585786437627+0j)	0.653739612	0.64
588	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
584	TRUE	5	1	0.094737	0.552925039	(0.518805695908+0j)	0.385487528	0.49
576	TRUE	5	2	0.138298	0.34322638	(0.518805695908+0j)	0.875	0.25
588	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
578	TRUE	6	1	0.142857	0.662976797	(0.485863070665+0j)	0.125	0.64
578	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
587	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
576	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
584	TRUE	4	2	0.156863	0.600995247	(0.585786437627+0j)	1.037037037	0.49
582	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
582	TRUE	5	2	0.141509	0.329538833	(0.38196601125+0j)	1.625	0.25
581	TRUE	5	2	0.138298	0.34322638	(0.518805695908+0j)	0.875	0.25
580	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
577	TRUE	5	4	0.12963	0.267480847	(0.518805695908+0j)	2.033057851	0.16
583	TRUE	7	3	0.133333	0.162096162	(0.267949192431+0j)	2.111570248	0.09
593	TRUE	5	4	0.132653	0.325259063	(0.38196601125+0j)	1.125	0.25
578	TRUE	5	1	0.122222	0.586885112	(0.38196601125+0j)	0.625	0.49
575	TRUE	7	2	0.116505	0.293576612	(0.38196601125+0j)	1.0752	0.25
581	TRUE	5	4	0.174757	0.356937409	(0.38196601125+0j)	2.132963989	0.25
578	TRUE	4	2	0.161905	0.691812857	(0.585786437627+0j)	0.653739612	0.64
590	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
588	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64

580	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
576	TRUE	5	3	0.122449	0.325259063	(0.38196601125+0j)	1.125	0.25
579	TRUE	4	3	0.166667	0.600995247	(0.585786437627+0j)	1.037037037	0.49
584	TRUE	6	2	0.111111	0.230633378	(0.267949192431+0j)	2.081632653	0.16
584	TRUE	5	1	0.103774	0.50729257	(0.518805695908+0j)	0.475206612	0.49
587	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
581	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
585	TRUE	6	3	0.145455	0.16844995	(0.324869129433+0j)	2.181818182	0.09
577	TRUE	4	4	0.176471	0.600995247	(0.585786437627+0j)	1.037037037	0.49
574	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
586	TRUE	4	3	0.166667	0.600995247	(0.585786437627+0j)	1.037037037	0.49
585	TRUE	5	4	0.132653	0.325259063	(0.38196601125+0j)	1.125	0.25
578	TRUE	4	2	0.156863	0.600995247	(0.585786437627+0j)	1.037037037	0.49
586	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
594	TRUE	5	2	0.141509	0.329538833	(0.38196601125+0j)	1.625	0.25
586	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
576	TRUE	7	2	0.117647	0.150416403	(0.321719649958+0j)	1.677083333	0.09
582	TRUE	6	3	0.098214	0.277780859	(0.38196601125+0j)	1	0.25
571	TRUE	6	2	0.14433	0.171488913	(0.324869129433+0j)	1.918367347	0.09
581	TRUE	5	2	0.141509	0.329538833	(0.38196601125+0j)	1.625	0.25
588	TRUE	4	2	0.156863	0.600995247	(0.585786437627+0j)	1.037037037	0.49
572	TRUE	5	3	0.163043	0.304347826	(1+0j)	0.75	0.16
594	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64

592	TRUE	6	2	0.14433	0.171488913	(0.324869129433+0j)	1.918367347	0.09
580	TRUE	4	2	0.156863	0.600995247	(0.585786437627+0j)	1.037037037	0.49
578	TRUE	4	6	0.196078	0.600995247	(0.585786437627+0j)	1.037037037	0.49
584	TRUE	5	3	0.259259	0.632203883	(0.38196601125+0j)	0.578512397	0.64
587	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
579	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
582	TRUE	4	1	0.14433	0.875815036	(0.585786437627+0j)	0.299168975	0.81
581	TRUE	5	2	0.155963	0.403164156	(0.38196601125+0j)	1.179138322	0.36
577	TRUE	5	2	0.160377	0.433312418	(0.38196601125+0j)	1.625	0.36
575	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
581	TRUE	5	5	0.184466	0.356937409	(0.38196601125+0j)	2.132963989	0.25
579	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
578	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
585	TRUE	6	3	0.131313	0.305692087	(0.485863070665+0j)	0.737240076	0.25
582	TRUE	5	2	0.166667	0.464840444	(0.518805695908+0j)	1.375	0.36
590	TRUE	6	1	0.085106	0.303964424	(0.267949192431+0j)	0.991735537	0.25
577	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
581	TRUE	4	2	0.238938	0.684306562	(0.585786437627+0j)	1.008310249	0.64
590	TRUE	6	2	0.14433	0.171488913	(0.324869129433+0j)	1.918367347	0.09
577	TRUE	3	3	0.21875	0.465753425	(1+0j)	2.407407407	0.04
572	TRUE	4	1	0.14433	0.875815036	(0.585786437627+0j)	0.299168975	0.81
580	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
569	TRUE	4	6	0.196078	0.600995247	(0.585786437627+0j)	1.037037037	0.49

572	TRUE	6	1	0.111111	0.281846292	(0.267949192431+0j)	0.737240076	0.25
581	TRUE	6	2	0.108911	0.303235039	(0.438447187191+0j)	0.873345936	0.25
575	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
594	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
586	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
593	TRUE	4	9	0.22549	0.600995247	(0.585786437627+0j)	1.037037037	0.49
580	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
576	TRUE	4	4	0.176471	0.600995247	(0.585786437627+0j)	1.037037037	0.49
578	TRUE	5	4	0.174757	0.356937409	(0.38196601125+0j)	2.132963989	0.25
572	TRUE	5	3	0.122449	0.325259063	(0.38196601125+0j)	1.125	0.25
580	TRUE	5	1	0.103774	0.50729257	(0.518805695908+0j)	0.475206612	0.49
577	TRUE	4	2	0.139785	0.358762193	(0.585786437627+0j)	1.440443213	0.25
570	TRUE	5	8	0.202128	0.34322638	(0.518805695908+0j)	0.875	0.25
578	TRUE	6	2	0.12	0.211803511	(0.267949192431+0j)	1.438016529	0.16
584	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
588	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
588	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
575	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
575	TRUE	5	4	0.132653	0.325259063	(0.38196601125+0j)	1.125	0.25
586	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
572	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
570	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
579	TRUE	5	2	0.224299	0.300756645	(0.38196601125+0j)	1.065759637	0.25

586	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
585	TRUE	5	1	0.103774	0.50729257	(0.518805695908+0j)	0.475206612	0.49
583	TRUE	4	2	0.161905	0.691812857	(0.585786437627+0j)	0.653739612	0.64
569	TRUE	5	1	0.103774	0.50729257	(0.518805695908+0j)	0.475206612	0.49
585	TRUE	4	1	0.181818	0.869955872	(0.585786437627+0j)	0.387811634	0.81
587	TRUE	5	2	0.112245	0.325259063	(0.38196601125+0j)	1.125	0.25
580	TRUE	5	4	0.22549	0.592291425	(0.518805695908+0j)	1.375	0.49
581	TRUE	6	1	0.085106	0.303964424	(0.267949192431+0j)	0.991735537	0.25
579	TRUE	10	1	0.054348	0.118402613	(0.139194146888+0j)	1.734693878	0.09
582	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
586	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
556	TRUE	6	3	0.1	0.211803511	(0.267949192431+0j)	1.438016529	0.16
582	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
585	TRUE	4	6	0.196078	0.600995247	(0.585786437627+0j)	1.037037037	0.49
585	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
575	TRUE	6	3	0.147059	0.156711707	(0.38196601125+0j)	1.677083333	0.09
580	TRUE	5	1	0.128713	0.533556089	(0.38196601125+0j)	0.725623583	0.49
577	TRUE	5	3	0.150943	0.329538833	(0.38196601125+0j)	1.625	0.25
594	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
584	TRUE	5	1	0.094737	0.552925039	(0.518805695908+0j)	0.385487528	0.49
569	TRUE	5	1	0.103774	0.50729257	(0.518805695908+0j)	0.475206612	0.49
586	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
575	TRUE	4	3	0.166667	0.600995247	(0.585786437627+0j)	1.037037037	0.49

586	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
574	TRUE	4	4	0.176471	0.600995247	(0.585786437627+0j)	1.037037037	0.49
577	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
586	TRUE	4	1	0.14433	0.875815036	(0.585786437627+0j)	0.299168975	0.81
585	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
584	TRUE	5	2	0.155963	0.403164156	(0.38196601125+0j)	1.179138322	0.36
581	TRUE	5	1	0.139785	0.714949111	(0.518805695908+0j)	0.272108844	0.64
580	TRUE	5	1	0.103774	0.50729257	(0.518805695908+0j)	0.475206612	0.49
552	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
580	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
583	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
590	TRUE	5	1	0.139785	0.714949111	(0.518805695908+0j)	0.272108844	0.64
571	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
592	TRUE	4	1	0.181818	0.869955872	(0.585786437627+0j)	0.387811634	0.81
585	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
583	TRUE	3	2	0.269231	0.525641026	(1+0j)	0.65625	0.36
582	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
573	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
592	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
576	TRUE	4	4	0.176471	0.600995247	(0.585786437627+0j)	1.037037037	0.49
576	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
580	TRUE	5	2	0.11215	0.427603244	(0.518805695908+0j)	1.065759637	0.36
576	TRUE	5	1	0.103774	0.50729257	(0.518805695908+0j)	0.475206612	0.49

588	TRUE	5	2	0.141509	0.329538833	(0.38196601125+0j)	1.625	0.25
578	TRUE	4	2	0.188119	0.353960396	(1+0j)	0.476454294	0.25
581	TRUE	4	3	0.166667	0.600995247	(0.585786437627+0j)	1.037037037	0.49
571	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
576	TRUE	4	3	0.166667	0.600995247	(0.585786437627+0j)	1.037037037	0.49
590	TRUE	5	2	0.112245	0.325259063	(0.38196601125+0j)	1.125	0.25
576	TRUE	4	4	0.176471	0.600995247	(0.585786437627+0j)	1.037037037	0.49
572	TRUE	4	1	0.141509	0.29245283	(1+0j)	0.24	0.25
590	TRUE	4	1	0.130435	0.765691857	(0.585786437627+0j)	0.543209877	0.64
592	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
584	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
596	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
580	TRUE	5	6	0.153061	0.325259063	(0.38196601125+0j)	1.125	0.25
577	TRUE	4	4	0.32381	0.649376246	(0.585786437627+0j)	1.813148789	0.49
580	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
580	TRUE	8	3	0.087912	0.215072679	(0.277406895678+0j)	1.3184	0.16
592	TRUE	5	1	0.139785	0.714949111	(0.518805695908+0j)	0.272108844	0.64
576	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
586	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
584	TRUE	4	5	0.186275	0.600995247	(0.585786437627+0j)	1.037037037	0.49
581	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
585	TRUE	4	1	0.181818	0.869955872	(0.585786437627+0j)	0.387811634	0.81
579	TRUE	4	3	0.166667	0.600995247	(0.585786437627+0j)	1.037037037	0.49

579	TRUE	5	3	0.219048	0.549675578	(0.518805695908+0j)	0.952380952	0.49
571	TRUE	6	2	0.101852	0.21596245	(0.267949192431+0j)	2.033057851	0.16
582	TRUE	3	2	0.210526	0.653939886	(1+0j)	1.259515571	0.49
588	TRUE	6	2	0.162162	0.234335232	(0.438447187191+0j)	1.553875236	0.16
582	TRUE	5	2	0.157895	0.24	(1+0j)	0.385487528	0.16
588	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
581	TRUE	7	1	0.096154	0.258979093	(0.260322690258+0j)	0.538461538	0.25
576	TRUE	4	2	0.156863	0.600995247	(0.585786437627+0j)	1.037037037	0.49
575	TRUE	7	4	0.111111	0.145314926	(0.260322690258+0j)	2.1875	0.09
573	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
588	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
577	TRUE	5	5	0.142857	0.325259063	(0.38196601125+0j)	1.125	0.25
581	TRUE	4	8	0.215686	0.600995247	(0.585786437627+0j)	1.037037037	0.49
576	TRUE	4	4	0.176471	0.600995247	(0.585786437627+0j)	1.037037037	0.49
574	TRUE	5	2	0.112245	0.325259063	(0.38196601125+0j)	1.125	0.25
585	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
582	TRUE	3	2	0.269231	0.525641026	(1+0j)	0.65625	0.36
578	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
581	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
583	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
585	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
582	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
584	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81

586	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
578	TRUE	5	1	0.10101	0.536618514	(0.38196601125+0j)	0.612244898	0.49
586	TRUE	3	1	0.207921	0.402640264	(1+0j)	0.14532872	0.36
583	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
573	TRUE	4	5	0.186275	0.600995247	(0.585786437627+0j)	1.037037037	0.49
584	TRUE	5	1	0.122222	0.586885112	(0.38196601125+0j)	0.625	0.49
584	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
583	TRUE	4	5	0.186275	0.600995247	(0.585786437627+0j)	1.037037037	0.49
589	TRUE	3	1	0.202128	0.475177305	(1+0j)	0.3046875	0.36
568	TRUE	4	1	0.12381	0.691812857	(0.585786437627+0j)	0.653739612	0.64
587	TRUE	5	3	0.150943	0.329538833	(0.38196601125+0j)	1.625	0.25
592	TRUE	4	3	0.272727	0.306753507	(0.585786437627+0j)	1.855955679	0.16
572	TRUE	4	2	0.188119	0.353960396	(1+0j)	0.476454294	0.25
585	TRUE	4	1	0.174419	0.975917816	(0.585786437627+0j)	0.24691358	0.81
567	TRUE	4	1	0.14433	0.875815036	(0.585786437627+0j)	0.299168975	0.81

Table C.3: Computation results on $\hat{G}$s which do not satisfied theorem 2.2.2

$\hat{G}_n$ [1]	H_b^G [2]	H_n^G [3]	s^TQs	ν	$\frac{\epsilon \frac{n^2}{y^Ts}+\alpha\theta}{n+\theta}$	α	θ	ϵ
591	TRUE	6	5	0.23	0.186149062	(0.438447187191+0j)	3	0.04
584	TRUE	6	4	0.17308	0.155081226	(0.324869129433+0j)	3.36	0.04
570	TRUE	16	5	0.13208	0.085600818	(0.150331459494+0j)	7.474048	0.04
568	TRUE	8	5	0.12621	0.09642005	(0.238442818168+0j)	2.5472	0.04
576	TRUE	8	6	0.28421	0.119405531	(0.318669356395+0j)	3.03155	0.04
586	TRUE	6	4	0.17308	0.155081226	(0.324869129433+0j)	3.36	0.04
575	TRUE	5	4	0.30612	0.259019445	(0.518805695908+0j)	2.82	0.09
583	TRUE	3	1	0.27835	0.264604811	(1+0j)	0.020761	0.25
580	TRUE	6	5	0.2233	0.196776673	(0.38196601125+0j)	2.408163	0.09
571	TRUE	9	6	0.17476	0.078612387	(0.186230723133+0j)	2.444444	0.04
592	TRUE	7	5	0.17708	0.140571084	(0.38196601125+0j)	2.666667	0.04
590	TRUE	7	5	0.14	0.109544934	(0.225377100496+0j)	3.123967	0.04
588	TRUE	3	1	0.27835	0.264604811	(1+0j)	0.020761	0.25
584	TRUE	7	6	0.14679	0.118873016	(0.267949192431+0j)	3.096408	0.04
586	TRUE	5	4	0.30189	0.239384855	(0.518805695908+0j)	3.179012	0.04
577	TRUE	6	3	0.24719	0.202348897	(0.485863070665+0j)	2.875346	0.04
579	TRUE	8	5	0.17925	0.156811451	(0.277406895678+0j)	6.016529	0.04
583	TRUE	7	4	0.15625	0.114495038	(0.260322690258+0j)	2.719008	0.04

[1] Number of nodes in $\hat{G}$

[2] Is the subgraph H of G bipartite?

[3] Number of nodes in H^G

583	TRUE	6	5	0.18269	0.155081226	(0.324869129433+0j)	3.36	0.04
585	TRUE	6	6	0.19231	0.155081226	(0.324869129433+0j)	3.36	0.04
587	TRUE	6	4	0.17308	0.155081226	(0.324869129433+0j)	3.36	0.04
590	TRUE	6	4	0.21296	0.180003744	(0.38196601125+0j)	2.033058	0.09
583	TRUE	9	6	0.16364	0.119189674	(0.198062264195+0j)	6.46875	0.04
582	TRUE	6	4	0.17308	0.155081226	(0.324869129433+0j)	3.36	0.04
583	TRUE	7	7	0.28713	0.154649843	(0.260322690258+0j)	2.355388	0.09
577	TRUE	7	3	0.13043	0.116902406	(0.295532185741+0j)	2.31405	0.04
582	TRUE	11	7	0.21359	0.091931397	(0.206021483244+0j)	3.819263	0.04
574	TRUE	6	4	0.14019	0.139094541	(0.324869129433+0j)	2.734694	0.04
575	TRUE	13	4	0.11458	0.06677337	(0.139880203735+0j)	2.84375	0.04
569	TRUE	4	3	0.21053	0.203860797	(0.585786437627+0j)	1.755102	0.04
582	TRUE	8	8	0.19588	0.121092739	(0.250882452253+0j)	3.73535	0.04
581	TRUE	8	4	0.14141	0.12123063	(0.24340174614+0j)	3.977316	0.04
562	TRUE	7	5	0.16964	0.141391251	(0.398320868117+0j)	2.527778	0.04
581	TRUE	8	7	0.19626	0.126521406	(0.198062264195+0j)	7.383743	0.04
587	TRUE	8	10	0.21569	0.190844016	(0.198062264195+0j)	1.656805	0.16
582	TRUE	7	4	0.11881	0.09634671	(0.225377100496+0j)	2.355388	0.04
586	TRUE	7	4	0.17143	0.156308112	(0.295532185741+0j)	4.666667	0.04
579	TRUE	6	3	0.16346	0.134648428	(0.267949192431+0j)	3.36	0.04
575	TRUE	13	5	0.24	0.072788319	(0.106562037619+0j)	5.777778	0.04
583	TRUE	7	3	0.16667	0.130698111	(0.260322690258+0j)	3.933884	0.04
574	TRUE	9	4	0.14	0.105545902	(0.120614758428+0j)	1.331633	0.09

578	TRUE	6	5	0.26786	0.144169911	(0.267949192431+0j)	4.08	0.04
585	TRUE	8	5	0.17	0.110247087	(0.250882452253+0j)	3.111111	0.04
592	TRUE	7	4	0.15625	0.131671569	(0.321719649958+0j)	2.719008	0.04
586	TRUE	7	4	0.17143	0.156308112	(0.295532185741+0j)	4.666667	0.04
583	TRUE	5	5	0.19444	0.191318984	(0.38196601125+0j)	3.933884	0.04
576	TRUE	7	8	0.20952	0.142224314	(0.260322690258+0j)	4.666667	0.04
573	TRUE	7	4	0.12844	0.118873016	(0.267949192431+0j)	3.096408	0.04
586	TRUE	7	5	0.17308	0.146290652	(0.321719649958+0j)	3.528926	0.04
586	TRUE	8	4	0.10204	0.090267663	(0.186393497352+0j)	2.888889	0.04
574	TRUE	7	3	0.12	0.109544934	(0.225377100496+0j)	3.123967	0.04
590	TRUE	8	5	0.13636	0.110181146	(0.213682265271+0j)	4.222222	0.04
588	TRUE	7	4	0.17143	0.117320144	(0.198062264195+0j)	4.666667	0.04
578	TRUE	10	6	0.17308	0.075056093	(0.148665376528+0j)	3.265306	0.04
583	TRUE	8	4	0.11	0.092190179	(0.186393497352+0j)	3.111111	0.04
577	TRUE	9	6	0.14423	0.097290945	(0.211785863843+0j)	3.461538	0.04
578	TRUE	13	8	0.20792	0.094673331	(0.153190854822+0j)	7.296076	0.04
583	TRUE	8	4	0.16981	0.146672641	(0.253786811873+0j)	6.016529	0.04
579	TRUE	6	11	0.24038	0.155081226	(0.324869129433+0j)	3.36	0.04
568	TRUE	7	8	0.19792	0.131671569	(0.321719649958+0j)	2.719008	0.04
577	TRUE	8	4	0.14141	0.106174717	(0.198062264195+0j)	3.977316	0.04
579	TRUE	6	7	0.16	0.124352169	(0.267949192431+0j)	3.123967	0.04
584	TRUE	7	4	0.25743	0.239414565	(0.321719649958+0j)	2.355388	0.16

Table C.4: Computation results on $\hat{G}$ from Yeast protein-protein interaction network which satisfied the theorem 2.2.2

$\hat{G}_n$ [1]	H_b^G [2]	H_n^G [3]	s^TQs	ν	$\frac{\epsilon\frac{n^2}{y^Ts}+\alpha\theta}{n+\theta}$	α	θ	ϵ
2224	TRUE	8	1	0.068965517	0.168103448	(1+0j)	0.248888889	0.16
2224	TRUE	7	1	0.08	0.194285714	(1+0j)	0.24852071	0.16
2224	TRUE	7	1	0.112244898	0.510596605	(0.726927136932+0j)	0.103550296	0.49
2224	TRUE	4	1	0.141509434	0.29245283	(1+0j)	0.24	0.25
2224	TRUE	4	1	0.141509434	0.29245283	(1+0j)	0.24	0.25
2224	TRUE	4	1	0.141509434	0.29245283	(1+0j)	0.24	0.25
2224	TRUE	4	1	0.193548387	0.888289384	(0.585786437627+0j)	0.121883657	0.81
2224	TRUE	5	4	0.18627451	0.464840444	(0.518805695908+0j)	1.375	0.36
2224	TRUE	5	10	0.239130435	0.304347826	(1+0j)	0.75	0.16
2224	TRUE	3	15	0.396039604	0.857673555	(1+0j)	17.6736	0.04
2224	TRUE	4	3	0.333333333	0.395112016	(1+0j)	1.440443213	0.16
2224	TRUE	3	5	0.336633663	0.613861386	(1+0j)	1.04	0.36

[1] Number of nodes in $\hat{G}$

[2] Is the subgraph H of G bipartite?

[3] Number of nodes in H^G

Table C.5: Computation results on $\hat{G}$ from Yeast protein-protein interaction network which satisfied the theorem 2.2.2

$\hat{G}_n$ [1]	H_b^G [2]	H_n^G [3]	s^TQs	ν	$\frac{\epsilon\frac{n^2}{y^Ts}+\alpha\theta}{n+\theta}$	α	θ	ϵ
2224	TRUE	5	1	0.172414	0.144828	(1+0j)	0.034722	0.16
2224	TRUE	7	12	0.238532	0.192908	(0.295532185741+0j)	10.1024	0.04
2224	TRUE	8	9	0.264706	0.191302	(0.373801931473+0j)	5.487603	0.04
2224	TRUE	3	1	0.278351	0.264605	(1+0j)	0.020761	0.25
2224	TRUE	7	9	0.300971	0.20688	(0.38196601125+0j)	6.48	0.04
2224	TRUE	3	1	0.278351	0.264605	(1+0j)	0.020761	0.25
2224	TRUE	3	1	0.278351	0.264605	(1+0j)	0.020761	0.25
2224	TRUE	4	7	0.320755	0.286913	(0.585786437627+0j)	3.179012	0.04
2224	TRUE	6	11	0.356436	0.231372	(0.267949192431+0j)	2.244898	0.16
2224	TRUE	7	9	0.366667	0.27619	(1+0j)	2.11157	0.04
2224	TRUE	6	13	0.408602	0.227148	(0.38196601125+0j)	5.584775	0.04
2224	TRUE	6	10	0.425926	0.277045	(0.438447187191+0j)	8.75	0.04
2224	TRUE	4	9	0.460674	0.367978	(1+0j)	0.927336	0.16
2224	TRUE	3	3	0.460674	0.362996	(1+0j)	0.927336	0.16
2224	TRUE	5	5	0.446602	0.310478	(0.518805695908+0j)	2.132964	0.16

[1] Number of nodes in Ĝ

[2] Is the subgraph H of G bipartite?

[3] Number of nodes in H^G

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