Kolmogorov's Theorems

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Overview

- Full name: **Andrey Nikolaevich Kolmogorov**
- 20^{th} century Soviet mathematician
- Known for: Probability theory (axiomatic foundations), topology, Mathematical analysis, classical mechanics, algorithmic information theory, computational complexity and many more

Once his one of the students Vladimir Arnold has said:

"Kolmogorov - Poincaré - Gauss - Euler - Newton, are only five lives separating us from the source of our science"

http://en.wikipedia.org/wiki/Andrey_Kolmogorov

Kolmogorov's Theorems

- **Statistics**
  - Kolmogorov-Smirnov test
- **Probability theory**
  - Hahn-Kolmogorov theorem
  - Kolmogorov existence theorem
  - Kolmogorov continuity theorem
  - Kolmogorov's three-series theorem
  - Kolmogorov's zero-one law

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  - Kolmogorov's zero-one law →Tail event has probability of either 0 or 1

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Kolmogorov's Theorems cont.

- **Probability theory**
  - Chapman-Kolmogorov equations
  - Kolmogorov's inequality
- **Functional analysis**
  - Fréchet-Kolmogorov theorem

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Here, we will elaborate the followings:

- Kolmogorov's existence theorem
- Kolmogorov's continuity theorem

Reason: Fundamental to Stochastic process

Axiomatic Theory of Probability

σ- algebra

Let Ω be a set. A σ- algebra on Ω is a collection of subsets $\mathcal{F}$of Ω which obeys the following properties:

1. $\phi \in \mathcal{F}$
2. If $F \in \mathcal{F}$, then also $F^c \in \mathcal{F}$
3. If $A_1, A_2, ... \in \mathcal{F}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$

$(\Omega, \mathcal{F})$ is called a measurable space. A probability measure P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ s.t.

1. $P(\phi) = 0, P(\Omega) = 1$
2. If $A_1, A_2, ... \in \mathcal{F}$, and $\{A_i\}_{i=1}^{\infty}$ are pairwise disjoint then

$$P\left(\bigcup_{i \in 1} A_i\right) = \sum_{i \in I} P(A_i)$$

Axiomatic Theory of Probability cont.

- The triple $(\Omega, \mathcal{F}, P)$ is called a probability space
- The subsets F of Ω which belong to $\mathcal{F}$ are called $\mathcal{F}$-measurable sets. In a probability context these sets are called *events*
- Given any family $\mathcal{U}$of subsets of Ω there is a smallest σ-algebra algebra $\mathcal{H}_{\mathcal{U}}$ containing $\mathcal{U}$, namely

$$\mathcal{H}_{\mathcal{U}} = \bigcap \{\mathcal{H}; \mathcal{H}\sigma - \text{algebra of } \Omega, \mathcal{U} \subset \mathcal{H}\}$$

Mathematicaly, $\mathcal{H}_{\mathcal{U}}$ is the σ-algebra generated by $\mathcal{U}$

Example

Let, $\mathcal{U} =$ collection of all open subsets of a topological space Ω (e.g. $\Omega = \mathcal{R}^n$), then $\mathcal{B} = \mathcal{H}_{\mathcal{U}}$ is called the Borel σ-algebra on Ω and the elements $B \in \mathcal{B}$ are called Borel sets. $\mathcal{B}$ contains all open sets, all closed sets, all countable unions of closed sets, all countable intersections of such countable unions etc.

$\mathcal{F}$-measurable function

If $(\Omega, \mathcal{F}, P)$ is a given probability space, then a function $Y : \Omega \to \mathcal{R}^n$ is called $\mathcal{F}$-measurable if

$$Y^{-1}(U) = \{\omega \in \Omega; Y(\omega) \in U\} \in \mathcal{F}$$

For all open sets $U \in \mathcal{R}^n$ (or, equivalently, for all Borel sets $U \subset \mathcal{R}^n$)

A **random variable** X is an $\mathcal{F}$-measurable function $X : \Omega \to \mathcal{R}^n$. Every random variable induces a probability measure μ_X on $\mathcal{R}^n$, defined by $\mu_X(B) = P(X^{-1}(B))$. μ_X is called the distribution of X

Stochastic Process

Definition

A stochastic process is parameterized collection of random variables

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathcal{R}^n$

T can be $[0, \infty)$ or $[a, b]$ or non-negative integers or subsets of $\mathcal{R}^n, n \geq 1$ Now, if we fix $t \in T$, then we have a random variable

$$\omega \to X_t(\omega); \omega \in \Omega$$

If we fix $\omega \in \Omega$, we have the path of X_t:

$$t \to X_t(\omega); t \in T$$

Stochastic Process cont.

Finite-dimensional distributions

A finite dimensional distributions of the process $\{X_t\}_{t \in T}$ are the measures $\mu_{t_1, t_2, .., t_k}$ defined on $\mathcal{R}^{nk}$, $k = 1, 2, ...$ by

$$\mu_{t_1, t_2, ..., t_k}(F_1 \times F_2 \times \cdots \times F_k) = P[X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k]; \quad t_i \in T$$

$\{F_j\}_{j=1}^{k}$ are Borel sets in $\mathcal{R}^n$

Stochastic Process cont.

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Question: Given a family $\{\nu_{t_1,t_2,..,t_k}; k \in \mathcal{N}_+, t_i \in T\}$ of probability measures on $\mathcal{R}^{nk}$, can we construct a stochastic process $Y = \{Y_t\}_{t \in T}$ having $\nu_{t_1,t_2,..,t_k}$ as its FDD?

Stochastic Process cont.

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A finite dimensional distributions of the process $\{X_t\}_{t\in T}$ are the measures $\mu_{t_1,t_2,..,t_k}$ defined on $\mathcal{R}^{nk}$, $k = 1, 2, \ldots$ by

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Question: Given a family $\{\nu_{t_1,t_2,..,t_k}; k \in \mathcal{N}_+, t_i \in T\}$ of probability measures on $\mathcal{R}^{nk}$, can we construct a stochastic process $Y = \{Y_t\}_{t\in T}$ having $\nu_{t_1,t_2,..,t_k}$ as its FDD?
Answer: Kolmogorov Existence Theorem

Kolmogorov Existence/Extension Theorem

Theorem

For all $t_1, t_2, .., t_k \in T, k \in \mathcal{N}_+$, let $\nu_{t_1,t_2,..,t_k}$ be the probability measures on $\mathcal{R}^{nk}$ s.t.

$$\nu_{t_1,..,t_k}(F_1 \times \cdots \times F_k) = \nu_{t_{\sigma(1)},..,t_{\sigma(k)}}(F_{\sigma(1)} \times \cdots \times F_{\sigma(k)})$$

for all permutations σ on $\{1, 2, ..., k\}$ and

$$\nu_{t_1,..,t_k}(F_1 \times \cdots \times F_k) = \nu_{t_1,..,t_k,...,t_{k+m}}(F_{\sigma(1)} \times \cdots \times F_{\sigma(k)} \times \mathcal{R}^n, ..., \times \mathcal{R}^n)$$

for all $m \in \mathcal{N}_+$. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{X_t\}_{t \in T}$ on $\Omega, X_t : \Omega \to \mathcal{R}^n$, s.t.

$$\nu_{t_1,..,t_k}(F_1 \times \cdots \times F_k) = P[X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k], \forall t_i \in T, k \in \mathcal{N}_+$$

for all Borel sets F_i

Example: Brownian Motion

Brownian Motion: Stochastic process $B_t(\omega)$, interpreted as the position at time t of the pollen grain ω

Now to construct such process, we need to construct a family of appropriate probability measures $\{\nu_{t_1,t_2,..,t_k}\}$ according to Kolomogorov's way. Fix $x \in \mathcal{R}^n$ and define

$$p(t,x,y) = (2\pi t)^{-n/2} \exp(-\frac{(x-y)^2}{2t}) \text{ for } y \in \mathcal{R}^n, t > 0$$

Now if, $0 \leq t_1 \leq t_2 \cdots \leq t_k$ define a measure $\nu_{t_1,t_2,..,t_k}$ on $\mathcal{R}^{nk}$ by

$$\int_{F_1 \times \cdots \times F_k} p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\dots p(t_k-t_{k-1},x_{k-1},x_k)dx_1 \dots dx_k$$

Now, it is possible to generate a permutation σ such that $0 \leq \sigma(t_1) \leq \sigma(t_2) \cdots \leq \sigma(t_k)$, therefore, satisfies the first condition automatically. Now, consider the notation $dy = dy_1 \dots dy_k$ for Lebesgue measure and $\int_{\mathcal{R}^n} p(t,x,y) = 1 \forall t \geq 0 \implies$ second condition is also satisfied.

Kolmogorov Continuity Theorem

Observations from Brownian motion example

- The Brownian motion thus defined is not unique.
- Paths may not be continuous (esp. if the Borel sets are intervals) which should be.

The second observation motivates us to identify $\omega \in \Omega$ with a continuous function from $[0, \infty)$ to $\mathcal{R}^n$. The continuity of path is assured by **Kolmogorov Continuity Theorem**

Theorem

Suppose that the process $X = \{X_t\}_{t \geq 0}$ satisfies the following condition: $\forall T > 0 \exists (\alpha, \beta, D) > 0$ s.t

$$E[|X_t - X_s|^\alpha] \leq D|t - s|^{1+\beta}; \ \ 0 \leq s, t \leq T$$

Then there exists a continuous version of X

Example: Brownian Motion

For the Brownian motion we can have:

$$E[|B_t - B_s|^4] \leq n(n+2)|t-s|^2$$

Therefore, $\alpha = 4, D = n(n+2), \beta = 1$ and we have a continuous version.

References

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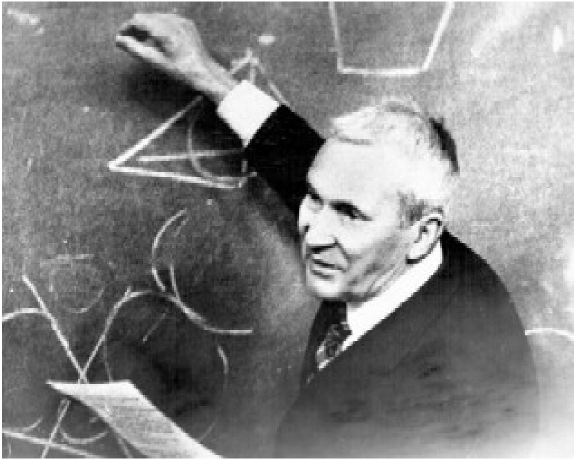

April 25, 1903 - October 20, 1987

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