Identifying large bipartite subgraphs of a graph: combinatorial versus spectral approaches

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Bipartite graphs have important applications in various fields of science and technology.

Examples

- Decode code words received from the channel (Factor graphs and Tanner graphs)
- Petri nets (Directed bipartite graphs)
- Movies preferences: How much someone would enjoy a movie based on their preferences.
- . . .

In fact it has been shown that *all complex networks can be viewed as bipartite structures sharing some important statistics..*

-Guillaume, J. L., & Latapy, M. (2004), *Info. proc. lett.*, **90**(5)

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ON SOME EXTREMAL PROBLEMS IN GRAPH THEORY

BY
P. ERDÖS

ABSTRACT

The author proves that if C is a sufficiently large constant then every graph of n vertices and $[Cn^{3/2}]$ edges contains a hexagon $X_1, X_2, X_3, X_4, X_5, X_6$ and a seventh vertex Y joined to X_1, X_3 and X_5. The problem is left open whether our graph contains the edges of a cube, (i.e. an eight vertex Z joined to X_2, X_4 and X_6).

Lemma

Every $G(n; m)$ has an even subgraph having at least $m/2$ edges.

Making a graph bipartite

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Lemma

Every $G(n; m)$ has an even subgraph having at least $m/2$ edges.

For a simple, connected graph $G(V, E)$ having a degree matrix D, the three characteristic matrices indicate bipartivity in the following way:

- Adjacency matrix (A)
  - G is bipartite $\implies \lambda^{A}_{\text{min}} = -\lambda^{A}_{\text{max}}$
  - G is non-bipartite, then λ^{A}_{min} is nearer to 0
- Normalized Laplacian matrix ($\mathcal{NL} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}}$, $L = D - A$)
  - G is bipartite $\iff \lambda^{\mathcal{NL}}_{\text{max}} = 2$
- Signless Laplacian matrix ($Q = D + A$)
  - G is bipartite $\iff \lambda^{Q}_{\text{min}} = 0$

Other measure:

$$\beta(G) = \frac{\#\text{even closed walks}}{\text{Total}\,\#\,\text{closed walks}} = \frac{\sum_{i=1}^{N} cosh(\lambda_i)}{\sum_{i=1}^{N} \exp(\lambda_i)}, \lambda_i \in \lambda_A$$

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Strengths and weaknesses

Matrix	Bipartite	# Components	# Bipartite Components	# Edges
Adjacency	Yes	No	No	Yes
Laplacian	No	Yes	No	Yes
Signless Laplacian	No	No	Yes	Yes
Normalized Laplacian	Yes	Yes	Yes	No

– Butler. S and Chung. F., Handbook of Linear Algebra, 2nd Eds. 2013

A **No** indicates the existence of two non-isomorphic graphs which have the same spectrum but differ in the indicated structure

Examples

- Multiplicity of $\lambda Q = 0$ indicates the $\#$ of bipartite components
- $\#$ of bipartite components is $\alpha + n - \sum_{i=1} \lambda_i^{\mathcal{NL}}$, α is the multiplicity of 2

- How can we find large bipartite subgraphs in G using the argument by Erdös and the spectrum of A, L, $\mathcal{NL}$, and Q?

- How efficient (in terms of size) are those methods in finding large subgraphs?

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1. Finding the largest bipartition in $G(V, E)$
   - Combinatorial approach
   - Spectral approaches
   - Initial results
2. New measures
   - Results with new measures
3. Summary and outlook

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2. New measures

3. Summary and outlook

1. partition V into two disjoint sets $(V_1 \cup V_2)$
2. transfer a vertex $v_i \in V_j$ to $V_{k \neq j}, j, k = \{1, 2\}$ if

$$\frac{2 * \deg(v_i) \text{in } V_j}{deg(v_i)} > 1$$

3. terminate if there is no more new movement.

Input: A simple, connected graph $G = (E, V)$

Result: $\frac{|E_{bipart}(G)|}{|E(G)|}$

Initialize: $V_{final} = \Phi$; $X \subset V$ and $Y = V \backslash \{X\}$

```
repeat
    if S_start = X(or Y) then
        if 2E_u^X > E_u^V ∧ u ∉ V_final (2E_v^Y > E_v^V ∧ v ∉ V_final) then
            Y ← u (X ← v);
            V_final ← u (V_final ← v)
        S_start ← Y (X)
until There is no movement of vertices;
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$E_i^H := \{|E| \; | i \in H \wedge (\forall j \in H, (i, j) \in E)\}$

Note: To minimize the effect of random partitioning, we consider the average of 10 different partitions.

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Combinatorial approach

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- Eigenvector based approach
  1. extract the non-zero entries of the eigenvectors corresponding to
     - $\lambda_{\min}^{A}$ and $\lambda_{\min}^{Q}$
     - $\lambda_{\max}^{\mathcal{NL}}$ and $\lambda_{\max}^{L}$
  2. make the bipartition based on the signs of the non-zero entries.
- Approach based on Estrada and Rodríguez-Velázquez's $\beta(G)$
  1. $e \leftarrow \arg\min_{e \in E} 1 - [\beta(G-e) - \beta(G)]$
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Graph Models	Parameter(s)	initial value : step size :final value
Erdös-Rényi (E-R)	p	0.1 : 0.1 : 1
Watts-Strogatz (W-S)	(β, k)	(0.3, 1 : 1 : 9)
Barabási-Albert (B-A)	n	1 : 1 : 10

- p: Probabilty of attachment
- (β, k): probability of rewiring and the mean degree
- n: number of edges to attach in every step

We generate 1000 different graphs with 20 vertices corresponding to each values in the respective model parameters.

Choice of graph models

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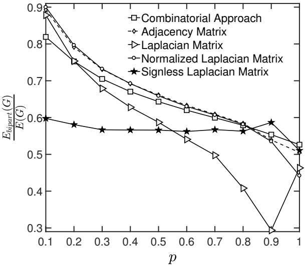

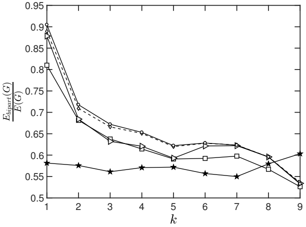

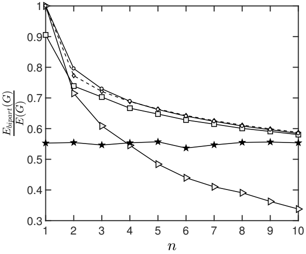

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- Performances of $\mathcal{NL}$ and A matrices are comparable even slightly better than that of the combinatorial approach.

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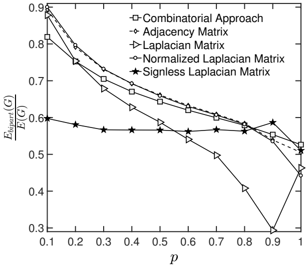

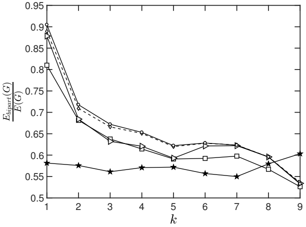

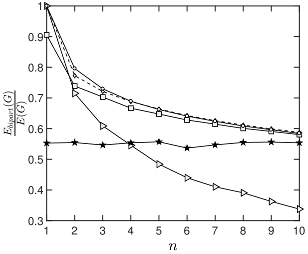

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1. Finding the largest bipartition in $G(V, E)$

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   - Results with new measures

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Let, Spectrum of $A(G) = (\boldsymbol{\nu}, \boldsymbol{\lambda})$
$\boldsymbol{\nu}$:= set of eigenvectors
$\boldsymbol{\lambda}$:= set of eigenvalues$=\{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V(G)|}\}$

- A matrix based approach
  1. $e \leftarrow \arg\max_{e \in E(G)} \frac{\nu^i_{\lambda_{|V(G)|}} \times \nu^j_{\lambda_{|V(G)|}}}{\nu^i_{\lambda_1} \times \nu^j_{\lambda_1}}$
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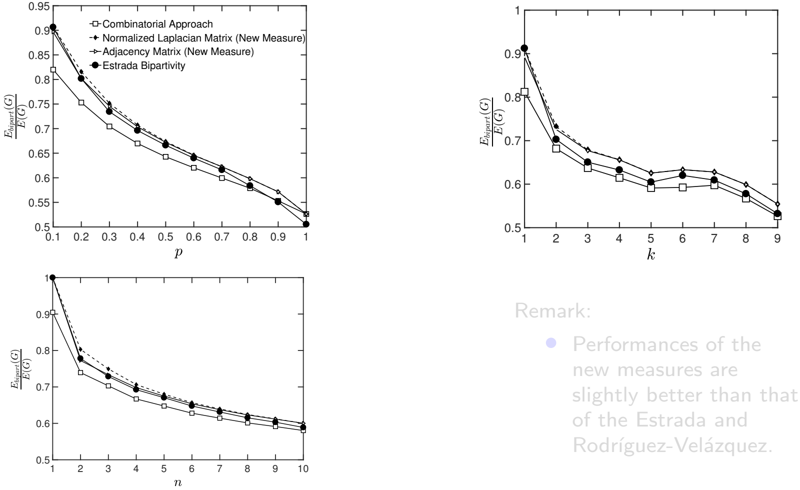

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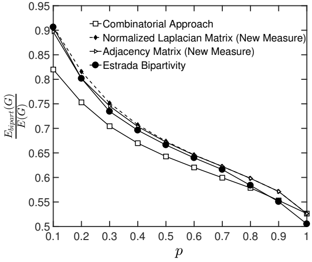

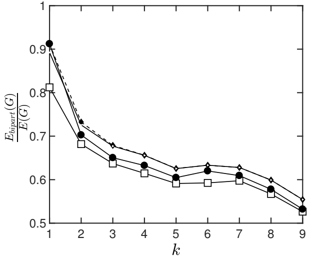

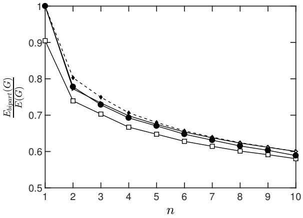

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