Stochastic processes: translate maths to sense

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Overview

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A collection of random variables indexed by time

$$Z(t) = \begin{cases} \{X_0, X_1, X_2, \dots\} & \text{Discrete time} \\ \{X_t\}_{t \geq 0} & \text{Continuous time} \end{cases}$$

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Example

1. $f(t) = t$ with prob. 1
2. $f(t) = \begin{cases} t \quad \forall t & \text{with prob.}0.5 \\ -t \; \forall t & \text{with prob.}0.5 \end{cases}$
3. For each t, $f(t) = \begin{cases} t \quad \forall t & \text{with prob.}0.5 \\ -t \; \forall t & \text{with prob.}0.5 \end{cases}$

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3. What is the probability of a boundary (rare) event (how often something extreme happens)?

$$\mathbb{P}(Z(t) > \text{high level})$$

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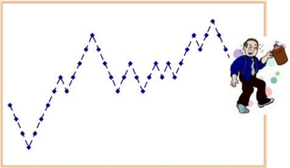
Simple Random Walk: simplest stochastic process



Definition

Let Y_i be i.i.d random variables such that:

$$Y_i = \begin{cases} 1 & \text{with prob.0.5} \\ -1 & \text{with prob.0.5} \end{cases}$$

and

$$X_t = \sum_{i=1}^{t} Y_i \ , X_0 = 0$$

Then $Z(t) = \{X_0, X_1, X_2, \dots\}$ is a one dimensional simple random walk

Simple Random Walk: properties

1. Expectation: $E(X_t) = 0$
2. Independent increments: For $0 = t_0 \leq t_1 \leq \cdots \leq t_k$, $\{X_{t_{i+1}} - X_{t_i}., i = 0, 1, 2 \ldots \; k-1\}$ are mutually independent
   Remark: what happens from time 20 to 30 is irrelevant to what happens from 40 to 50
3. (Stationary?) $\forall h \geq 1, t \geq 0$ the distribution of $X_{t+h} - X_t$ is same as the distribution of X_h
   Remark: If we look at the same amount of time, then what happens inside the time interval is irrelevant of starting point as the distribution is same.

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- Answer: $1/3$ (Martingale theory; comes later)

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- Mathematically:

$$\boxed{Pr(X_{t+1} = x | X_1 = x_1, \ldots, X_t = x_t) = Pr(X_{t+1} = x | X_t = x_t)}$$

if $Pr(X_1 = x_1, \ldots, X_t = x_t) > 0$, conditional probabilities are well defined

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- Master euqation for biochemical reactions $\rightarrow$ Chemical Master Equation (CME)- the fundamental equation for stochastic modelling of biochemical processes

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  - Main concept: If one plays a Martingale game, no matter what strategy he/she uses, the expected value will remain same.

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    Examples: In our coin toss game, the first time we win €100 or loose €50; time of first peak (not a stopping time ! why?)

Optional Stopping Time theorem

Theorem

(Informal way) Let there is a martingale with stopping time T, and suppose there is a constant K such that $T \leq K$ always. In this case the expectation value at the stopping time is always equal to the value at the beginning.

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Coin toss revisited

- consider the second situation: I will play until I win €100 or loose €50

Now, we can define a stopping time for this (strategy) say τ. According to the above theorem:

$$E(X_\tau) = X_0 = 0 \implies 100p + (1-p) * (-50) = 0 \implies p = 1/3$$

Brownian motion: simplest continuous time stochastic process

Denoted by $B(t)$ - serves basic model for the cumulative noise.

Definition

$B(t)$ is a stochastic process having the following properties:

- For any $0 < u < s < t$, $\{B(t) - B(s)\}$ and $\{B(u) - B(0)\}$ are independent of each other (Independent increments)
- $\{B(t) - B(s)\} \sim \mathcal{N}(0, t - s)$
- B(t) is continuous everywhere.

Path properties of $B(t)$

- Has infinite variation on any interval
- Quadratic variation on $[0, t]$ equal to t

Overview

Quadratic variation: invitation to Itô calculus

Definition

If g is a function of real variable, define its quadratic variation over the interval $[0, t]$ as the limit (when it exists)

$$\boxed{[g](t) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} (g(t_i^n) - g(t_{i-1}^n)^2}$$

Where the limit is taken over the partitions: $0 = t_0^n < t_1^n, \ldots, < t_n^n = t$, with $\delta_n = \max_{1 \leq i \leq n}(t_i^n - t_{i-1}^n)$

Important facts:

- For enough smooth function having finite variation, this quadratic variation is zero! $\implies$ We can apply Riemannian calculus.
- For B(t), quadratic variation is not zero hence we can not apply classical calculus $\rightarrow$ starting point of Itô calculus

Glimpse of Itô calculus: differentiation

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- But we can not do that! reason: quadratic variation of $B(t)$ is $dB(t)^2 = dt$
- How about $df = f'(B(t))dB(t)$? WRONG!
- Using Taylor expansion,

$$f(t+x) - f(t) = f'(t)x \text{ (classical calculus)}$$

Now for $f(B(t))$

$$f(B(t+x)) - f(B(t)) = f'(B(t))dB(t) + 1/2 f''(B(t))(dB(t))^2 + \ldots$$
$$df = f'(B(t))dB(t) + 1/2 f''(B(t))dt$$

This is Itô's lemma

Glimpse of Itô calculus: integration

Integrations is nothing but 'inverse' of differentiation
So Let us define:

$$F(t, B(t)) = \int f(t, B(t))dB(t) + \int g(t, B(t))dt$$

if $dF = fdB(t) + gdt$

- **Question**: Does there exists Riemannian sum type description?

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- **Reason**:Due to quadratic variance the two limits(left and right) are different (variance accumulates)!

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- Utility: convert a non-martingale process (with non-zero drift term) (might be risky) to a martingale process (safe!)

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References

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